

# Twisting cochains and twisted complexes

Simplicial methods in complex-analytic geometry

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24/07/19

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# Plan

History

Twisting cochains (OTT)

The bicomplex

The total complex

Maurer-Cartan

Strictification

Twisted complexes (BK)

Pretriangulated, triangulated, and stable

Generalisation of twisting cochains

Other fun things

The A-infinity Yoneda embedding

The bar construction

# History

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- John C Moore. “Differential homological algebra”. In: *Actes du Congrès International des Mathématiciens 1* (1970), pp. 335–339.

## Coherent sheaves

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# Nomenclature

## Lemma

*Let  $x \in \{\text{twisting, twisted}\}$  and  $y \in \{\text{cochain, complex}\}$ .*

*Then there exists somebody who uses “ $x y$ ” to denote what you call “ $x^c y^c$ ”.*

## Corollary

*Sometimes it can be hard to figure out what people mean.*

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*Sometimes it can be hard to figure out what people mean.*

I like “twisting cochain” for the topological Čech definition, which is a special case of a “twisted complex”, defined in the dg-categorical setting.

But that’s very possibly just me.

## Twisting cochains (OTT)

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## Definition (Stein spaces)

A complex-analytic<sup>1</sup> manifold  $Y$  is said to be *Stein* if it is

1. *holomorphically convex*; and
2. *holomorphically separable*.

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<sup>1</sup>analytic =  $\mathcal{O}_Y$  is holomorphic functions,  $Y$  has the  $\mathbb{C}^n$ -induced topology;  
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Throughout,  $X$  is a complex-analytic manifold with a nice<sup>2</sup> cover  $\mathcal{U} = \{U_\alpha\}_{\alpha \in I}$ .

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<sup>2</sup>Locally finite, Stein, and trivialising (for the bundles in question).

## Endomorphisms of bounded-graded modules

Let  $V = \{V_\alpha^\bullet\}$  be a collection of *bounded-graded*  $\mathcal{O}_{U_\alpha}$ -modules:

$$V_\alpha^\bullet = \bigoplus_{q \in \mathbb{N}} V_\alpha^q \quad \text{such that } V_\alpha^q \text{ is zero for all but finitely many } q.$$

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### Definition (Endomorphisms)

The collection of *degree- $q$  endomorphisms*  $\text{End}^q(V)$  of  $V$  is, over each  $U_{\alpha_0 \dots \alpha_p}$ , given by

$$\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = \bigoplus_{i \in \mathbb{Z}} \text{Hom}(V_{\alpha_p}^i|_{U_{\alpha_0 \dots \alpha_p}}, V_{\alpha_0}^{i+q}|_{U_{\alpha_0 \dots \alpha_p}}).$$



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We discuss this later.

# The deleted Čech complex

## Definition (Deleted Čech complex)

Define the chain complex  $(\hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V)), \hat{\delta})$  by

$$\hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) = \bigoplus_{(\alpha_0, \dots, \alpha_p)} \text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}}$$

(where  $\text{End}^q(V)|_{U_{\alpha_0 \dots \alpha_p}} = 0$  if  $U_{\alpha_0 \dots \alpha_p} = \emptyset$ ) with the **deleted Čech differential**

$$\hat{\delta}: \hat{\mathcal{C}}^p(\mathcal{U}, \text{End}^q(V)) \rightarrow \hat{\mathcal{C}}^{p+1}(\mathcal{U}, \text{End}^q(V))$$

$$(\hat{\delta}c)_{\alpha_0 \dots \alpha_{p+1}} = \sum_{i=1}^p (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

## A notational note

We use  $\hat{\mathcal{C}}$  and  $\hat{\delta}$  for the *deleted* Čech objects and  $\check{\mathcal{C}}$  and  $\check{\delta}$  for the 'full' Čech objects.

## Further structure

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- We could define the same complex for an arbitrary bounded graded vector bundle, i.e.

$$\hat{\mathcal{C}}^p(\mathcal{U}, V^q) = \bigoplus_{(\alpha_0, \dots, \alpha_p)} V_{\alpha_0}^q$$

but where the deleted Čech differential only omits the *first* index (but includes the  $(p+1)$ th).

## A brief interlude on bundles

A holomorphic vector bundle  $E$  on  $X$  is described exactly by its *transition maps*  $g_{\alpha\beta} \in \mathrm{GL}(n, \mathbb{C})$ , which describe the change in trivialisation from over  $U_\beta$  to over  $U_\alpha$ .



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2.  $g_{\alpha\alpha} = \mathrm{id}$  (the *invertibility* condition).

Note that these are maps from  $E|_{U_{\alpha_p}}$  to  $E|_{U_{\alpha_0}}$  in the specific case where  $p = 1$ .

## Rewriting the cocycle condition

Thinking of  $g_{\alpha\beta}$  as an element of  $\hat{\mathcal{C}}^1(\mathcal{U}, E)$ , we see that

$$\begin{aligned}(\hat{\delta}g)_{\alpha\beta\gamma} &= -g_{\alpha\gamma} \\(g \cdot g)_{\alpha\beta\gamma} &= g_{\alpha\beta}g_{\beta\gamma}.\end{aligned}$$

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This means that we can rewrite the cocycle condition as

$$\hat{\delta}g + g \cdot g = 0,$$

which looks like the *Maurer-Cartan equation* (an observation to which we will later return).

## Definition (Twisting cochains)

A (holomorphic) twisting cochain over  $V$  is a formal sum

$$a = \bigoplus_{k \in \mathbb{N}} a^{k, 1-k}$$

where  $a^{k, 1-k} \in \hat{\mathcal{C}}^k(\mathcal{U}, \text{End}^{1-k}(V))$  such that

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2.  $a_{\alpha\alpha}^{1,0} = \text{id}$ .

The invertibility condition “should” really be weakened by asking only that  $a_{\alpha\alpha}^{1,0}$  be *homotopic* to the identity.



## Twisting cochains (cont.)

### Warning

The multiplication is **not** simply component-wise: it is given by taking all possible combinations, i.e.

$$(a \cdot b)^{p,s} = \bigoplus_{\substack{q+q'=p \\ t+t'=s}} a^{q,t} \cdot b^{q',t'}$$

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- If  $V$  has a differential then  $a$  is an element of total degree 1.
- We haven't said *when* twisting cochains exist, but under pretty mild assumptions they always do (by an inductive construction).

## Unpacking the definition

$(k = 0) \rightsquigarrow a_{\alpha}^{0,1} \cdot a_{\alpha}^{0,1} = 0$ , which tells us that  $a_{\alpha}^{0,1}$  is a *differential* on  $V_{\alpha}^{\bullet}$ .

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$(k = 1) \rightsquigarrow a_{\alpha}^{0,1} \cdot a_{\alpha\beta}^{1,0} = a_{\alpha\beta}^{1,0} \cdot a_{\beta}^{0,1}$ , which tells us that we have a *chain map of chain complexes*

$$a_{\alpha\beta}^{1,0} : (V_{\beta}^{\bullet} | U_{\alpha\beta}, a_{\beta}^{0,1}) \rightarrow (V_{\alpha}^{\bullet} | U_{\alpha\beta}, a_{\alpha}^{0,1})$$

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$(k = 2) \rightsquigarrow -a_{\alpha\gamma}^{1,0} + a_{\alpha\beta}^{1,0} \cdot a_{\beta\gamma}^{1,0} = a_{\alpha}^{0,1} \cdot a_{\alpha\beta\gamma}^{2,-1} + a_{\alpha\beta\gamma}^{2,-1} \cdot a_{\gamma}^{0,1}$ , which says that  $a_{\alpha\beta\gamma}^{2,-1}$  witnesses a *chain homotopy* between  $a_{\alpha\gamma}^{1,0}$  and  $a_{\alpha\beta}^{1,0} \cdot a_{\beta\gamma}^{1,0}$ . On  $\alpha\beta\alpha$  and  $\beta\alpha\beta$  this tells us that  $a_{\alpha\beta}^{1,0}$  and  $a_{\beta\alpha}^{1,0}$  are *chain homotopic inverses*, i.e. *quasi-isomorphism*.

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### Extra-curricular

By taking (internal) homology we obtain something strict: a complex of *coherent sheaves*  $H^\bullet(a)$ . This is because quasi-isomorphisms become strict isomorphisms in homology.

We can use this fact to construct twisting cochains that resolve coherent sheaves by taking *local* resolutions by vector bundles.

# The total differential

## Lemma

For any  $a \in \text{Tot}^1 \hat{\mathcal{C}}^\bullet(\mathcal{U}, \text{End}^\circ(V))$ , the map

$$\begin{aligned} D_a: \text{Tot}^r \hat{\mathcal{C}}^\bullet(\mathcal{U}, V^\circ) &\rightarrow \text{Tot}^{r+1} \hat{\mathcal{C}}^\bullet(\mathcal{U}, V^\circ) \\ c &\mapsto \hat{\delta}c + c \cdot a \end{aligned}$$

defines a differential (i.e. squares to zero) if and only if  $a$  is a twisting cochain.

## Proof.

(Tedious) definition chasing. □

## The total differential (cont.)

We can actually define twisting cochains in a different way using this lemma (but we won't do so today).

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But this approach lets us think of a twisting cochain as a *first-order perturbation of the deleted Čech differential*.

## Examples

### Example

Look at the most trivial example: let  $V$  be an *ungraded* vector bundle, and  $a = a^{0,1} + a^{1,0}$ , where  $a_\alpha^{0,1} = \text{id}_{V_\alpha}$ , and the  $a^{1,0}$  are the transition maps. Then

$$(D_a c)_{\alpha_0 \dots \alpha_{p+1}} = a_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \dots \alpha_{p+1}} + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}.$$

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We can't use the full Čech differential on  $\hat{\mathcal{C}}^{\bullet}(U, V^{\circ})$  because everything has to lie over  $U_{\alpha_0}$ , but this total differential solves that problem —  $a_{\alpha_0 \alpha_1}^{1,0}$  is a (quasi-)isomorphism.

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We can't use the full Čech differential on  $\mathcal{E}^{\bullet}(U, V^{\circ})$  because everything has to lie over  $U_{\alpha_0}$ , but this total differential solves that problem —  $a_{\alpha_0 \alpha_1}^{1,0}$  is a (quasi-)isomorphism.

A spectral-sequence argument shows that, in fact,  $D_a$  here really is 'the same as' the full Čech differential.



## Examples (cont.)

### Example

Now look at a slightly-less trivial example: let  $V^\bullet$  consist of complexes  $(V_\alpha^\bullet, d_\alpha)$  of vector bundles, and  $a = a^{0,1} + a^{1,0}$ , where  $a_\alpha^{0,1} = d_\alpha$ , and the  $a^{1,0}$  are the transition maps. Then

$$\begin{aligned} (D_a c)_{\alpha_0 \dots \alpha_{p+1}} &= (-1)^p a_{\alpha_0}^{0,1} c_{\alpha_0 \dots \alpha_p} + a_{\alpha_0 \alpha_1}^{1,0} c_{\alpha_1 \dots \alpha_{p+1}} \\ &\quad + \sum_{i=1}^{p+1} (-1)^i c_{\alpha_0 \dots \hat{\alpha}_i \dots \alpha_{p+1}}. \end{aligned}$$

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Identifying the second and third terms with the full Čech differential, as before, gives the usual total differential of the Čech bicomplex:

$$D_a = \delta \pm d_V.$$

## Why this emphasis on the first index?

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- Transition maps naturally go from  $\alpha_p$  to  $\alpha_0$ .
- We want to be able to compare local things, and we need to pull everything back to lie over the same open set in order to do so.

# The Maurer-Cartan equation in other fields

Subject	Equation	Interpretation
Differential geometry	$F_{\nabla} = dA + A \cdot A$	curvature of a Koszul connection <sup>3</sup>
Gauge theory	$\Omega = dA + \frac{1}{2}[A \wedge A]$	curvature of a principal connection
Deformation theory	$\partial a + \frac{1}{2}[a, a]$	deformations of f.d. associative $k$ -algebras with unit <sup>4</sup>

<sup>3</sup>Here be Christoffel symbols.

<sup>4</sup>There is also the beautiful fact (that we won't explain at all) that  $\text{MC}(A \otimes \mathfrak{g}) \simeq \text{Hom}_{\text{dgAlg}}(\text{CE}(\mathfrak{g}), A)$ .

## Motto

Solutions to (i.e. zeros of) the Maurer-Cartan equation are always (in some sense) *flat objects*.

# Can we strictify?

No.<sup>5</sup>

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<sup>5</sup>But sort of, yes.

## Twisted complexes (BK)

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## Motto

*Pretriangulated dg-categories* are those whose homotopy category is *triangulated*, where *triangulated* denotes the structure left over from taking the homotopy category of a *stable*  $(\infty, 1)$ -category.<sup>6</sup>

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<sup>6</sup>Loop spaces and suspensions form an equivalence.

### Problem

There is no reason for an arbitrary dg-category to be pretriangulated, which means that homotopy theorists might be unhappy. This is bad.

## Stability (cont.)

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### Solution

Twisted complexes à la Bondal and Kapranov.

## Definition

A *dg-category* is a category enriched over chain complexes. That is, the hom-sets are hom-*complexes*.

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Let  $\mathcal{A}$  be a dg-category. Then a *twisted complex*  $\mathfrak{C}$  over  $\mathcal{A}$  is a collection

$$\mathfrak{C} = \{E_i \in \mathcal{A}, q_{ij}: E_i \rightarrow E_j \mid i, j \in \mathbb{Z}\}$$

such that

- all but finitely many of the  $E_i$  are zero;
- the  $q_{ij}$  are of degree  $i - j + 1$ ; and
- $dq_{ij} + \sum_{s \in \mathbb{Z}} q_{sj}q_{is} = 0$ .

# The main result

## Theorem

*Given a dg-category, the smallest dg-category containing it in which we can define shifts and functorial cones is exactly its category of twisted complexes.*

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*Further, if the original dg-category is pretriangulated then this embedding is a quasi-equivalence (which lets us pull back the shift and the cones, which descend exactly to a triangulated structure on the homotopy category).*

*We can also use the simplicial nerve to construct stable  $(\infty, 1)$ -categories.*



# Twisting cochains as twisted complexes

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It is **not** the case that, by picking the 'right' dg-category  $\mathcal{A}$ , we can recover the definition of holomorphic twisting cochains from that of twisted complexes.

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## Warning

It is **not** the case that, by picking the 'right' dg-category  $\mathcal{A}$ , we can recover the definition of holomorphic twisting cochains from that of twisted complexes.

Rather, twisting cochains are a *specific* case of twisted complexes.

## Constructing a twisted complex from a twisting cochain

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## Motto

Twisting cochains are twisted complexes that are *concentrated in degree zero* and *'projective/free'*.

## Other fun things

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Bernhard Keller. *Introduction to A-infinity algebras and modules*.  
2001. arXiv: 9910179v2 [math.RA].

# Preliminary definitions

## Definition

- An  $A_\infty$ -algebra is 'like a loop space' — it is a  $k$ -algebra with a graded derivation, but where associativity of multiplication holds only up to homotopy, which hold only up to homotopy, which...;
- $\mathcal{C}_\infty \mathcal{A}$  is the *category of  $A_\infty$ -modules* over an  $A_\infty$ -algebra  $\mathcal{A}$ ;
- $\mathcal{D}_\infty \mathcal{A}$  is the *homotopy category of  $\mathcal{C}_\infty \mathcal{A}$* ; and  $\text{tria} \mathcal{A}$  the ( $A_\infty$ -version of the) *triangulated subcategory of  $\mathcal{D} \mathcal{A}$*  generated by the free rank-1  $\mathcal{A}$  module.

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In the non- $A_\infty$  case, up to isomorphism, the objects of  $\text{tria} \mathcal{A}$  are bounded complexes of finitely-generated free  $\mathcal{A}$ -modules, and the morphisms "are" homotopy classes of morphisms of complexes.

# Factorisation of the Yoneda functor

## Theorem

Let  $\mathcal{A}$  be an  $A_\infty$ -category (with strict identities). Then the Yoneda functor  $\mathfrak{y}$  factors through the  $A_\infty$ -category  $\text{tw}\mathcal{A}$  of twisting cochains

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\mathfrak{y}_1} & \text{tw}\mathcal{A} \\ & \searrow \mathfrak{y} & \downarrow \mathfrak{y}_2 \\ & & \mathcal{C}_\infty\mathcal{A} \end{array}$$

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Further,  $\mathfrak{y}_1$  is (strictly) fully faithful, and  $\mathfrak{y}_2$  induces an equivalence

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This lets us formalise what the  $A_\infty$ -version of  $\text{tw}\mathcal{A}$  really is.

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Note that this *is* Maurer-Cartan, since the last term is the product  $\tau \star \tau$  in the *convolution algebra*.

## The cobar-bar adjunction

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We can do something similar to then turn  $\tau_f$  into a chain map (again, by Maurer-Cartan)  $f': \Omega C \rightarrow A$ .

## Koszul duality and twisted tensor products — full circle

Given some twisting cochain  $\tau: C \rightarrow A[1]$  we can define the *twisted tensor products*  $- \otimes_{\tau} A$  and  $- \otimes_{\tau} C$  on the level of (co)derived categories of (co)modules.

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These two functors form an equivalence if and only if

$$A \otimes_{\tau} C \otimes_{\tau} A \rightarrow A$$

is a quasi-isomorphism.

If this is the case then

$$\begin{aligned} H_{\bullet} C &= \mathrm{Tor}_{\bullet}^A(k, k) \\ H^{\bullet} A &= \mathrm{Ext}_{\bullet}^C(k, k). \end{aligned}$$