Holomorphic Deligne cohomology

For the traveller who wishes to see all the sights
(but doesn't really want to prove anything)

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Table of non-contents

- Deligne–Beilinson cohomology (logarithmic growth);
- Regulator maps (from algebraic K-theory to Deligne(–Beilinson) cohomology);
- Cycle classes (and normal crossing divisors);
- Secondary characteristic classes;
- Abstract differential cohomology (in a cohesive topos); and
- …a *lot* of other things.

There is a lot of interplay between *smooth*, *complex-algebraic*, and *complex-analytic* geometry. Many notions that can be defined in one setting can also be defined analogously in the other two, and some settings make certain notions much simpler.

One thing that is very hard in the complex-analytic setting is the notion of "finiteness" (more specifically, coherence of sheaves of $\mathscr{O}_X$-modules). In all three settings, "finite data" always exists *locally* — in the smooth setting, we can patch this local data together with partitions of unity; in the complex-algebraic setting, this local data is often "magically" global; in the complex-analytic setting …

Motivation

Given a complex-analytic manifold, the constant sheaf $\underline{\mathbb{C}}$ has two important structures:

- the integral structure, coming from $\underline{\mathbb{Z}} \hookrightarrow \underline{\mathbb{C}}$; and
- the Hodge structure, coming from the de Rham resolution $\Omega_X^\bullet \xrightarrow{\sim} \underline{\mathbb{C}}$.

The former describes *topological* information, and the latter describes *geometric* information.

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The former describes *topological* information, and the latter describes *geometric* information.

Deligne cohomology is a theory which contains *both* of these structures at once, and gives very powerful tools for understanding the intricate interplay between topological and geometric objects.

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…in fact, a little bit of topology goes a surprisingly long way.

The setting

- X : a paracompact complex-analytic manifold;
- $\mathscr{U}$: a Stein[1] open cover of X;
- $\mathscr{O}_X = \Omega^0_X$: the sheaf of holomorphic functions on X;
- $\Omega^\bullet_X$: the holomorphic de Rham complex on X.

However: everything that we *define* can be done in the algebraic or smooth setting, but the *properties* of the objects are then **very** different.

(Feel free to ask me, at any point, what the algebraic analogue of any statement is, and I will *try* to answer …)

[1] Think of Stein as the holomorphic analogue of contractible (for topological spaces). Important lemma: we can always refine any given open cover to a Stein subcover.

Reminder on sheaf cohomology

Notation

Various notions of cohomology:

- $\mathrm{H}^\bullet(X,-)$: values in an abelian group;
- $\mathbb{H}^\bullet(X,-)$: values in a (complex of) sheaves (cf. next slide);
- $\mathscr{H}^\bullet(-)$: internal cohomology of a cochain complex.

Čech cohomology

In this talk, we will always calculate sheaf cohomology via *Čech cohomology* (which we're allowed[2] to do, even without taking a limit over refinement of covers).

That is, given a complex of sheaves $\mathscr{F}^\bullet$ on X, we have that

$$\mathbb{H}^q(X,\mathscr{F}^\bullet) \cong \check{\mathrm{H}}^q(\mathscr{U},\mathscr{F}^\bullet) \coloneqq \mathscr{H}^q\,\mathrm{Tot}\,\check{\mathscr{C}}^\star(\mathscr{U},\mathscr{F}^\bullet)$$

where the Čech bicomplex $\check{\mathscr{C}}^\star(\mathscr{U},\mathscr{F}^\bullet)$ is

$$\check{\mathscr{C}}^p(\mathscr{U},\mathscr{F}^q) \coloneqq \prod_{\alpha_0<\ldots<\alpha_p} \mathscr{F}^q(U_{\alpha_0\ldots\alpha_p})$$

and where Tot gives the total complex of a bicomplex.

[2] We assume X to be paracompact, $\mathscr{U}$ is Stein, and the Ω_X^i are all coherent.

Reminder on connections

The Atiyah exact sequence

Let $(X, \mathscr{O}_X)$ be a ringed space, and E a locally free sheaf of $\mathscr{O}_X$-modules on X.

Definition

The *Atiyah exact sequence* (or *jet sequence*) of E is the short exact sequence of $\mathscr{O}_X$-modules

$$0 \to E \otimes_{\mathscr{O}_X} \Omega^1_X \to J^1(E) \to E \to 0$$

where $J^1(E) \coloneqq (E \otimes \Omega^1_X) \oplus E$ **as a $\underline{\mathbb{C}}$-module**, but with an $\mathscr{O}_X$-action given by

$$f(s \otimes \omega, t) = (fs \otimes \omega + t \otimes \mathrm{d}f,\ ft).$$

Connections on a locally free sheaf

There are a *lot* of ways of defining/thinking about/understanding connections, but we'll use the following.

Definition

A (*Koszul*) *connection* ∇ on E is a splitting of the Atiyah exact sequence of E. That is, a $\mathbb{C}$-linear morphism

$$\nabla\colon E \to E \otimes_{\mathscr{O}_X} \Omega^1_X$$

satisfying the *Leibniz rule*

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$$

for any local sections s of E, and f of $\mathscr{O}_X$.

Locally, any connection can be written as

$$\mathrm{d} + \overline{\omega}$$

where $\overline{\omega} \in \underline{\mathrm{Hom}}(E, E \otimes \Omega^1_X)$.

Note that, since E is locally free,

$$\underline{\mathrm{Hom}}(E, E \otimes \Omega^1_X) \cong \underline{\mathrm{Hom}}(E, E) \otimes \Omega^1_X$$

and so we can think of $\overline{\omega}$ as an *endomorphism-valued form*, i.e. a matrix of 1-forms.

By definition, connections are **not** $\mathscr{O}_X$-linear:

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f \neq f\nabla(s).$$

Curvature

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If we compose ∇ with itself, however, we do end up with something $\mathscr{O}_X$-linear:

$$\nabla^2 \colon E \to E \otimes \Omega_X^2$$

where, to make sense of this composition, we impose the "general" Leibniz rule

$$\nabla(s \otimes \omega) = \nabla(s) \wedge \omega + s \otimes \mathrm{d}\omega.$$

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Definition

We call $\kappa(\nabla) \coloneqq \nabla^2 \colon E \to E \otimes \Omega^2_X$ the *curvature* of ∇. Note that, **locally**, it can be written as $\mathrm{d}\overline{\omega} + \overline{\omega}^2$.

Chern–Weil theory

Long story short:

- we can use[3] the curvature to get Chern classes of E (that don't depend on the choice of ∇); **but** …

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- …unfortunately, in the complex-analytic case, global holomorphic connections almost never[4] exist; **but** …

[3] By looking at the trace of $\kappa(\nabla)^p$.

[4] Indeed, since the connection is exactly a splitting of the Atiyah exact sequence, the Atiyah class (the Ext class of the sequence) is exactly the obstruction towards the existence of a global connection, and the Atiyah class "is the same as" the first Chern class. So non-trivial first Chern class $\implies$ no global holomorphic connection exists.

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- …unfortunately, in the complex-analytic case, global holomorphic connections almost never[4] exist; **but** …
- …we can use simplicial constructions as a replacement for partitions of unity and make things all work out just fine.

---

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The Deligne complex and Deligne cohomology

The Deligne complex

We define[5] the constant sheaf $\underline{\mathbb{Z}(p)} := \underline{(2\pi i)^p\mathbb{Z}}$.

[5] The $(2\pi i)^p$ factor comes from the fact that we wish our Chern classes to be algebraic, and that there is some interplay between the topological and the holomorphic definitions. Really, this whole story comes from the fact that $\int_{|z|=1} \frac{\mathrm{d}z}{z} = 2\pi i \ldots$

The Deligne complex

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Definition

The (*holomorphic*) *Deligne complex* on X is

$$\begin{aligned}\mathbb{Z}(p)^\bullet_{\mathrm{Del}} &\coloneqq \underline{\mathbb{Z}(p)} \hookrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \ldots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X \\ &= \underline{\mathbb{Z}(p)}[0] \oplus \Omega^{\bullet \leqslant p-1}_X[-1].\end{aligned}$$

(As previously mentioned, we can define this in any setting where the notation makes sense.)

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The (p, q)*-th Deligne cohomology of* X is

$$\mathrm{H}^{q,p}_{\mathrm{Del}}(X) := \mathbb{H}^q(X, \mathbb{Z}(p)^\bullet_{\mathrm{Del}}).$$

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$$\mathrm{H}^{q,p}_{\mathrm{Del}}(X) := \mathbb{H}^q(X, \mathbb{Z}(p)^\bullet_{\mathrm{Del}}).$$

There is a (graded commutative) ring structure on $\bigoplus_{p,q} \mathrm{H}^{q,p}_{\mathrm{Del}}(X)$ induced by the multiplication

$$\begin{aligned}\mathbb{Z}(p)^\bullet_{\mathrm{Del}} \otimes \mathbb{Z}(p')^\bullet_{\mathrm{Del}} &\longrightarrow \mathbb{Z}(p+p')^\bullet_{\mathrm{Del}} \\ (x,y) &\longmapsto \begin{cases} x \cdot y & \text{if } \deg x = 0; \\ x \wedge \mathrm{d}y & \text{if } \deg x > 0 \textit{ and } \deg y = p'; \\ 0 & \text{otherwise.} \end{cases}\end{aligned}$$

Examples for small values of p and q

The vague idea

Very roughly:

- p is the amount of "(higher) geometry" that we see, and
- q is the cohomological degree.

We are generally interested in the cases where $p \leqslant q$, with particular interest in the specific cases $\mathrm{H}^{p,p}_{\mathrm{Del}}$ and $\mathrm{H}^{2p,p}_{\mathrm{Del}}$.

$p = 0$

Since $\mathbb{Z}(0)^\bullet_{\mathrm{Del}} = \underline{\mathbb{Z}}[0]$, we recover *singular cohomology*:

$$\mathrm{H}^{0,q}_{\mathrm{Del}}(X) \coloneqq \mathbb{H}^q(X, \underline{\mathbb{Z}}[0]) \cong \mathrm{H}^q(X, \mathbb{Z}).$$

$\mathbb{Z}(p)^\bullet_{\mathrm{Del}} \coloneqq \underline{\mathbb{Z}(p)} \hookrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X.$

$p = 1$

The complex $\mathbb{Z}(1)^\bullet_{\mathrm{Del}} = \underline{(2\pi i)\mathbb{Z}} \hookrightarrow \mathscr{O}_X$ is quasi-isomorphic to the complex $\mathscr{O}_X^\times[-1]$ via the *exponential exact sequence*

$$0 \to \underline{(2\pi i)\mathbb{Z}} \hookrightarrow \mathscr{O}_X \xrightarrow{\exp} \mathscr{O}_X^\times \to 0.$$

$\mathbb{Z}(p)^\bullet_{\mathrm{Del}} \coloneqq \underline{\mathbb{Z}(p)} \hookrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X.$

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We thus obtain an isomorphism

$$\mathrm{H}^{q,1}_{\mathrm{Del}}(X) \xrightarrow{\sim} \mathbb{H}^q(X, \mathscr{O}_X^\times[-1]) \cong \mathrm{H}^{q-1}(X, \mathscr{O}_X^\times).$$

$$\mathbb{Z}(p)^\bullet_{\mathrm{Del}} \coloneqq \underline{\mathbb{Z}(p)} \hookrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X.$$

$p = 2$

We have the quasi-isomorphism

$$\begin{tikzcd}
\underline{\mathbb{Z}(2)} \arrow[r, hook] \arrow[d] & \mathscr{O}_X \arrow[r, "\mathrm{d}"] \arrow[d, "\exp"] & \Omega^1_X \arrow[d, "\frac{-1}{2\pi i}\cdot(-)"] \\
0 \arrow[r] & \mathscr{O}_X^\times \arrow[r, "\mathrm{dlog}"'] & \Omega^1_X
\end{tikzcd}$$

which gives us the isomorphism

$$\mathrm{H}^{q,2}_{\mathrm{Del}}(X) \xrightarrow{\sim} \mathbb{H}^q\big(X, (\mathscr{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega^1_X)[-1]\big).$$

$\mathbb{Z}(p)^\bullet_{\mathrm{Del}} \coloneqq \underline{\mathbb{Z}(p)} \hookrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X.$

Summary, for $q = 2$

If we start thinking about *bundles* then we can start to formalise this idea that "*p controls the amount of geometry that we see*". So let's fix $q = 2$ and list the results.

Summary, for $q = 2$

- $\mathrm{H}^{2,0}_{\mathrm{Del}}(X) \cong \mathrm{H}^2(X, \mathbb{Z})$
  Smooth principal $\mathbb{C}^\times$-bundles (i.e. *smooth* complex line bundles) on X;

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  Isomorphism classes of holomorphic principal $\mathbb{C}^\times$-bundles (i.e. *holomorphic* line bundles) with (global) *holomorphic* connections;
- $\mathrm{H}^{2,q}_{\mathrm{Del}}(X)$ for $q \geqslant 3$
  Isomorphism classes of holomorphic principal $\mathbb{C}^\times$-bundles (i.e. *holomorphic* line bundles) with *flat* (global) *holomorphic* connections.

Chern classes

It is a classical fact[6] that a holomorphic line bundle on X admits a (global) holomorphic connection if and only if its first Chern class is trivial.

This fits with our interpretation of Deligne cohomology, by looking at the exact sequence

$$\underbrace{\mathbb{H}^1(X, \mathscr{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega_X^1)}_{\mathrm{H}^{2,2}_{\mathrm{Del}}(X)} \longrightarrow \underbrace{\mathbb{H}^1(X, \mathscr{O}_X^\times)}_{\mathrm{H}^{2,1}_{\mathrm{Del}}(X)} \xrightarrow{c_1} \mathbb{H}^1(X, \Omega_X^1).$$

[6] An argument using the *Atiyah class*.

A small algebraic remark

In the complex-*algebraic* setting, *any* algebraic bundle can *always* be endowed with a (global) connection. This is reflected in the fact that that $\mathrm{H}^{2,1}_{\mathrm{algDel}}(Y) \cong \mathrm{H}^{2,2}_{\mathrm{algDel}}(Y)$.

More generally[7], $\mathrm{H}^{p,p}_{\mathrm{algDel}} \cong \mathrm{H}^{p+1,p}_{\mathrm{algDel}} \cong \ldots \cong \mathrm{H}^{2p,p}_{\mathrm{algDel}}$, which is most definitely *not* true in the complex-analytic setting.

[7] Caveat: I can't remember the proof of this, but I'm reasonably certain that it is true. I think it even follows from a quasi-isomorphism of complexes…

Recovering other Chern classes

Given a holomorphic line bundle $\mathscr{L}$ on X, we have the notion of the first Chern class $c_1(\mathscr{L})$ living in various cohomology theories:

- $c_1^{\mathrm{top}}(\mathscr{L}) \in \mathrm{H}^2(X, \mathbb{Z})$ in singular cohomology;
- $c_1^{\mathrm{tdR}}(\mathscr{L}) \in \mathbb{H}^1(X, \Omega_X^1)$ in Dolbeault/truncated de Rham cohomology; and
- $c_1^{\mathrm{dR}}(\mathscr{L}) \in \mathrm{H}^2_{\mathrm{dR}}(X)$ in de Rham cohomology.

All three of these can be recovered[8] from the first Chern class $c_1^D(\mathscr{L}) \in \mathrm{H}^{2,1}_{\mathrm{Del}}(X)$ in Deligne cohomology.

[8]Using the fact that the image of a class in $\mathrm{H}^{2,1}_{\mathrm{Del}}(X)$ under dlog lives in $F^1\,\mathrm{H}^2_{\mathrm{dR}}(X)$, which is a subgroup of $\mathrm{H}^2_{\mathrm{dR}}(X)$, but also projects onto $\mathbb{H}^1(X, \Omega_X^1)$; we recover singular cohomology from the connecting morphism in the LES associated to the exponential exact sequence.

The bundle with connection given by multiplication

The multiplication

$$\mathbb{Z}(1)^{\bullet}_{\mathrm{Del}} \otimes \mathbb{Z}(1)^{\bullet}_{\mathrm{Del}} \to \mathbb{Z}(2)^{\bullet}_{\mathrm{Del}}$$

thus allows us to construct, given any two functions $f, g \in \mathbb{H}^0(X, \mathscr{O}_X^{\times})$, a holomorphic line bundle $r(f, g)$ with connection ∇.

Lemma

Both $r(f, g) \otimes r(g, f)$ *and* $r(1 - f, f)$ *are isomorphic to the trivial line bundle with trivial connection* $(\mathscr{O}_X, \mathrm{d})$.

We won't mention this again here, but it is useful!

Other definitions of the Deligne complex

The “good” definition[9] of the Deligne complex is as the mapping cone

$$\mathbb{Z}(p)^{\bullet}_{\mathrm{Del},\Delta} \coloneqq \operatorname{cone}(\underline{\mathbb{Z}} \oplus \Omega_X^{\bullet \geqslant p} \to \Omega^{\bullet})[-1].$$

[9] We’ll drop the $(2\pi i)^p$ factor for now, just for notational simplicity.

The mapping cone and the exponential exact sequence

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$$\mathbb{Z}(p)^\bullet_{\mathrm{Del},\Delta} \coloneqq \operatorname{cone}(\underline{\mathbb{Z}} \oplus \Omega_X^{\bullet \geqslant p} \to \Omega^\bullet)[-1].$$

But the exponential exact sequence also gives us the quasi-isomorphism

$$\begin{aligned} \mathbb{Z}(p)^\bullet_{\mathrm{Del}} &\xrightarrow{\sim} \Omega^{\bullet<p}(\mathrm{dlog})[-1] \\ &\coloneqq (\mathscr{O}_X^\times \xrightarrow{\mathrm{dlog}} \Omega_X^1 \to \ldots \to \Omega_X^{p-1})[-1] \end{aligned}$$

which turns out to be very useful as well.

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The mapping cone

The mapping cone gives us the diagram

$$\begin{tikzcd}
& \underline{\mathbb{Z}} \hookrightarrow \mathscr{O}_X \xrightarrow{\mathrm{d}} \dots \xrightarrow{\mathrm{d}} \Omega^{p-1}_X \arrow[ddl, "\mathrm{id}_{\mathbb{Z}}"'] \arrow[ddr, "\mathrm{d}"] & \\
& \mathbb{Z}(p)^{\bullet}_{\mathrm{Del},\Delta} \arrow[u, "\text{q-iso}"] \arrow[dl] \arrow[dr] & \\
\underline{\mathbb{Z}} \arrow[dr, hook] & & \Omega^{\bullet\geqslant p}_X \arrow[dl, hook] \\
& \underline{\mathbb{C}} \xrightarrow{\sim} \Omega^{\bullet}_X &
\end{tikzcd}$$

that commutes *up to homotopy* (since the composition of two arrows in a distinguished triangle is *homotopic to* zero).

Differential cohomology (briefly)

The idea behind differential cohomology

The "n-POV" of Deligne cohomology is that it is just one specific cocycle model for a more abstract theory, called *differential cohomology*[10], which can be defined as some functors $\mathrm{H}^\bullet_{\mathrm{diff}}$ into groups such that we have two short exact sequences (which further fit together into the *differential cohomology hexagon*).

I like to think of this as a sort of "motivic" style approach: we simply posit the existence of some natural transformations into other known cohomology theories and ask that they behave nicely.

[10] Or, more precisely, *ordinary* differential cohomology, since it is an enrichment of ordinary (i.e. Betti, or singular) integral cohomology.

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Some other models for differential cohomology include *Cheeger–Simons differential characters* (or *secondary characteristic classes*) and *bundle gerbes* (or *circle n-bundles*) *with connection*.

[10] Or, more precisely, *ordinary* differential cohomology, since it is an enrichment of ordinary (i.e. Betti, or singular) integral cohomology.

The two natural transformations

We require the existence of two natural transformations:

1. the *characteristic class* $c\colon \mathrm{H}^\bullet_{\mathrm{diff}}(-) \Rightarrow \mathrm{H}^\bullet(-,\mathbb{Z})$; and
2. the *curvature* $F\colon \mathrm{H}^\bullet_{\mathrm{diff}}(-) \Rightarrow \Omega^\bullet_{\mathrm{cl}}(-)$.

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2. the *curvature* $F\colon \mathrm{H}^\bullet_{\mathrm{diff}}(-) \Rightarrow \Omega^\bullet_{\mathrm{cl}}(-)$.

The former recovers the first Chern class of a line bundle in degree 2, and the Dixmier–Douady class of a bundle gerbe in degree 3; the latter actually lands in closed differential forms with *integral* periods (i.e. for all $\gamma\colon S^n \to X$, we have $\int_{S^n} \gamma^*\omega \in \mathbb{Z} \subset \mathbb{R}\mathbb{R}$).

The two short exact sequences

We also require the natural transformations to give us two short exact sequences:

1. the *characteristic class exact sequence*

$$0 \to \Omega^{n-1}(X)/\Omega^{n-1}_{\mathrm{int}}(X) \to \mathrm{H}^n_{\mathrm{diff}}(X) \xrightarrow{c} \mathrm{H}^n(X,\mathbb{Z}) \to 0\,;$$

2. the *curvature exact sequence*

$$0 \to \mathrm{H}^{n-1}(X,\mathrm{U}(1)) \to \mathrm{H}^n_{\mathrm{diff}}(X) \xrightarrow{F} \Omega^n_{\mathrm{int}}(X) \to 0.$$

(As we mentioned above, these two sequences fit together in a lovely hexagon, but we're not going to talk about that.)

Bundle gerbes

To talk about *bundle gerbes*, we really need to talk about *principal 2-bundles*, and thus about *Lie 2-groups*, and thus about *2-groups* …

…but this would take more time than I currently have, so let's just be vague!

Nomenclature warning: bundle gerbes are neither special cases nor generalisations of gerbes[11] (but they are somehow related).

[11] "Recall" that a gerbe is a sheaf of categories.

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Definition (…sort of)

Bundle gerbes are geometric objects that realise the 3-dimensional cohomology of a manifold.

[11] "Recall" that a gerbe is a sheaf of categories.

The analogy

Up to an abuse of nomenclature (restricting to only certain subclasses of the objects in questions), we have the following analogy from [Mur08]:

[Mur08] Michael K Murray, “An Introduction to Bundle Gerbes”. arXiv:0712.1651v3 [math.DG].

There are basically three ways of thinking about U(1) bundles over a manifold M:

(1) A certain kind of locally free sheaf on M.

(2) A co-cycle $g_{\alpha\beta} \colon U_\alpha \cap U_\beta \to \mathrm{U}(1)$ for some open cover $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ of M.

(3) A principal U(1) bundle $P \to M$.

[12]Rapidly: a surjective submersion $Y \to M$ and a U(1)-bundle $P \to Y^{[2]} \subset Y^2$ with an associative multiplication isomorphism $\pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$.

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(3) A principal U(1) bundle $P \to M$.

In the case of gerbes over M we can think of these as:

(1) A certain kind of sheaf of groupoids on M.

(2) A co-cycle $g_{\alpha\beta\gamma} \colon U_\alpha \cap U_\beta \cap U_\gamma \to \mathrm{U}(1)$ for some open cover $\mathscr{U} = \{U_\alpha \mid \alpha \in I\}$ of M or alternatively a choice of U(1) bundle $P_{\alpha\beta} \to U_{\alpha\beta}$ for each double overlap.

(3) A bundle gerbe.[12]

[12] Rapidly: a surjective submersion $Y \to M$ and a U(1)-bundle $P \to Y^{[2]} \subset Y^2$ with an associative multiplication isomorphism $\pi_3^*(P) \otimes \pi_1^*(P) \to \pi_2^*(P)$.

Intermediate Jacobians

Recall that the *Jacobian variety* of an algebraic curve X is the connected component of the identity of the *Picard variety* of X. Dual (as abelian varieties) to the Picard variety is the *Albanese variety* of X, which is the “free abelianisation”.

These two constructions (the Jacobian and the Albanese) appear as the extreme cases of a more general construction: that of the *intermediate Jacobian*.

The intermediate Jacobian

Definition

The *p-th intermediate Jacobian* $J^{p+1}(X)$ of a smooth projective complex variety X is, as a **real manifold**,

$$J^{p+1}(X) := \mathrm{H}^{2p+1}(X,\mathbb{R})/\,\mathrm{H}^{2p+1}(X,\mathbb{Z}).$$

This can be endowed with two[13] complex structures: the *Griffiths* and the *Weil*.

[13] The resulting complex manifolds are not isomorphic as complex manifolds, but *are* isomorphic as real symplectic manifolds.

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(More generally we can define the p-th intermediate Jacobian of any weight-$(2p+1)$ Hodge structure H as $J(H) = H_{\mathbb{C}}/(\mathrm{H}_{\mathbb{Z}} \oplus F^{p+1})$.)

[13] The resulting complex manifolds are not isomorphic as complex manifolds, but *are* isomorphic as real symplectic manifolds.

The relation to Deligne cohomology

Theorem

We have the short exact sequence of abelian groups

$$0 \to J^{2p-1}(X) \to \mathrm{H}^{2p,p}_{\mathrm{Del}}(X) \to \mathrm{H}^{p,p}_{\mathbb{Z}}(X) \to 0.$$

This follows, for example, from the short exact sequences of differential cohomology (along with some other lemmas).

Thank you for listening.