Simplicial connections and resolutions for coherent analytic sheaves

Tim Hosgood

2nd December, 2020

Centre for Quantum Technologies (Singapore)
Topos Institute (USA)

The main goal

Constructing characteristic classes of smooth vector bundles or coherent algebraic sheaves is “easy”.

The main goal

Constructing characteristic classes of smooth vector bundles or coherent algebraic sheaves is “easy”.

What about in the complex-analytic case?

Constructing characteristic classes of smooth vector bundles or coherent algebraic sheaves is "easy".

What about in the complex-analytic case?

That is, given some $(X, \mathcal{O}_X)$, where X is a paracompact complex-analytic manifold, and $\mathcal{O}_X$ is the sheaf of holomorphic functions, can we construct some nice

$$\mathsf{Coh}(X) \to \mathrm{H}^{\mathrm{even}}_{\mathrm{dR}}(X)$$

in a similar way?

Preliminaries

Coherent sheaves

Coherent sheaves are a generalisation of vector bundles that form an abelian category.

Coherent sheaves are a generalisation of vector bundles that form an abelian category.

Definition

A sheaf $\mathscr{E}$ on a ringed space $(X, \mathcal{O}_X)$ is said to be **of finite type** over $\mathcal{O}_X$ if, for all x in X, there exists some *surjective* morphism

$$\mathcal{O}_X^n|U \twoheadrightarrow \mathscr{F}|U$$

for some open neighbourhood U of x, and some $n \in \mathbb{N}$.

Definition

A sheaf $\mathscr{E}$ on a ringed space $(X, \mathcal{O}_X)$ is said to be **coherent** if

(i) $\mathscr{E}$ is of finite type over $\mathcal{O}_X$; and

(ii) the kernel of any $\mathcal{O}_X^n|U \to \mathscr{F}|U$ is of finite type.

Definition

The **Atiyah exact sequence** (or **jet sequence**) of a locally free sheaf E on a ringed space $(X, \mathcal{O}_X)$ is the short exact sequence

$$0 \to E \otimes_{\mathcal{O}_X} \Omega^1_X \to J^1(E) \to E \to 0$$

of $\mathcal{O}_X$-modules, where $J^1(E) = (E \otimes \Omega^1_X) \oplus E$ *as a* $\mathbb{C}_X$*-module,* but with an $\mathcal{O}_X$-action given by

$$f(s \otimes \omega, t) = (fs \otimes \omega + t \otimes \mathrm{d}f, ft).$$

The **Atiyah class** of E is the extension class

$$\mathrm{at}_E = [J^1(E)] \in \mathrm{Ext}^1_{\mathcal{O}_X}(E, E \otimes \Omega^1_X).$$

Connections

Definition

A **Koszul connection** ∇ on E is a splitting of the Atiyah exact sequence of E. That is, ∇ is a $\mathbb{C}$-linear morphism

$$\nabla\colon E \to E \otimes_{\mathcal{O}_X} \Omega^1_X$$

such that

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f$$

for any local section s of E.

Lemma

Locally, any connection can be written as

$$\mathrm{d} + \overline{\omega}$$

where $\overline{\omega} \in \underline{\mathrm{Hom}}(E, E \otimes \Omega^1_X) \cong \underline{\mathrm{Hom}}(E, E) \otimes \Omega^1_X$ *is an* ***endomorphism-valued form***.

By definition, connections are *not* $\mathcal{O}_X$-linear, i.e.

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f \neq f\nabla(s).$$

Curvature

By definition, connections are *not* $\mathcal{O}_X$-linear, i.e.

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f \neq f\nabla(s).$$

Definition

We say that a morphism $\widetilde{\nabla}\colon E \otimes \Omega_X^r \to E \otimes \Omega_X^{r+1}$ satisfies the **Leibniz rule** if

$$\widetilde{\nabla}(s \otimes \omega)\widetilde{\nabla}(s) \wedge \omega + s \otimes \mathrm{d}\omega.$$

By definition, connections are *not* $\mathcal{O}_X$-linear, i.e.

$$\nabla(fs) = f\nabla(s) + s \otimes \mathrm{d}f \neq f\nabla(s).$$

Definition

We say that a morphism $\widetilde{\nabla}\colon E \otimes \Omega_X^r \to E \otimes \Omega_X^{r+1}$ satisfies the **Leibniz rule** if

$$\widetilde{\nabla}(s \otimes \omega)\widetilde{\nabla}(s) \wedge \omega + s \otimes \mathrm{d}\omega.$$

Note that, if $\widetilde{\nabla}$ satisfies the Leibniz rule, then

$$\widetilde{\nabla}(fs \otimes \omega) = \widetilde{\nabla}(s \otimes f\omega).$$

Curvature

Definition

The **curvature** of a connection ∇ is the morphism

$$\kappa(\nabla) = \nabla \circ \nabla \colon E \to E \otimes \Omega_X^2$$

given by imposing the Leibniz rule.

Definition

The **curvature** of a connection ∇ is the morphism

$$\kappa(\nabla) = \nabla \circ \nabla \colon E \to E \otimes \Omega_X^2$$

given by imposing the Leibniz rule.

Lemma

Locally, the curvature of any connection can be written as

$$\mathrm{d}\overline{\omega} + \overline{\omega}^2.$$

How do all of the previous things link together? All of the above can be defined for *principal bundles*, and then we have the following theorem.

How do all of the previous things link together? All of the above can be defined for *principal bundles*, and then we have the following theorem.

Theorem (Chern–Weil homomorphism)

Let P be a G-principal bundle on X. Then there is a corresponding homomorphism

$$\mathbb{C}[\mathfrak{g}]^G \to \mathrm{H}^\bullet_{\mathrm{dR}}(X)$$

of $\mathbb{C}$-algebras given by evaluation on the curvature of any connection.

The "prototypical element" of $\mathbb{C}[\mathfrak{g}]^G$ is the **trace**.

Complex-analytic geometry

The algebraic setting

When $(X, \mathcal{O}_X)$ is a nice scheme over $\mathbb{C}$, then we can calculate the Chern classes of any coherent sheaf $\mathscr{E}$ on X by

1. taking a resolution $E^\bullet \to \mathscr{E}$ by locally free sheaves ; and
2. taking the alternating sum of $\mathrm{tr}\kappa(\nabla_i)^p$, where ∇_i is a connection on E^i.

Problems in the analytic setting

Both of the steps in the algebraic method go wrong in when $(X, \mathcal{O}_X)$ is a (paracompact) complex-analytic manifold:

1. coherent sheaves rarely admit resolutions by locally free sheaves ; and
2. locally free sheaves rarely admit connections.

Problems in the analytic setting

For the second point, note that the Atiyah class is exactly the obstruction towards admitting a (global) connection, by definition as an extension class.

For the second point, note that the Atiyah class is exactly the obstruction towards admitting a (global) connection, by definition as an extension class.

Lemma

The trace of the Atiyah class "is" the first Chern class. The traces of the higher Atiyah classes "are" the higher Chern classes/characters.

Problems in the analytic setting

For the second point, note that the Atiyah class is exactly the obstruction towards admitting a (global) connection, by definition as an extension class.

Lemma

The trace of the Atiyah class "is" the first Chern class. The traces of the higher Atiyah classes "are" the higher Chern classes/characters.

Combining these two things, we see that the only time we could maybe apply Chern–Weil theory is exactly when the characteristic classes are zero!

Local solutions to the problems

Lemma

Given a locally free sheaf E, we can always find a "nice" ***Stein*** *cover $\mathscr{U} = \{U_\alpha\}$ of X that trivialises E.*

Local solutions to the problems

Lemma

Given a locally free sheaf E, we can always find a "nice" ***Stein*** *cover $\mathscr{U} = \{U_\alpha\}$ of X that trivialises E.*

Lemma

Any coherent sheaf admits a ***local*** *resolution by locally free sheaves.*

Local solutions to the problems

Lemma

Given a locally free sheaf E, we can always find a "nice" ***Stein*** *cover $\mathscr{U} = \{U_\alpha\}$ of X that trivialises E.*

Lemma

Any coherent sheaf admits a ***local*** *resolution by locally free sheaves.*

Lemma

Any locally free sheaf on a ***Stein*** *manifold admits a holomorphic connection.*

Local solutions to the problems

Lemma

Given a locally free sheaf E, we can always find a "nice" ***Stein*** *cover $\mathscr{U} = \{U_\alpha\}$ of X that trivialises E.*

Lemma

Any coherent sheaf admits a ***local*** *resolution by locally free sheaves.*

Lemma

Any locally free sheaf on a ***Stein*** *manifold admits a holomorphic connection.*

So everything works *locally*.

Local solutions to the problems

Lemma

If we have a Stein cover $\mathscr{U} = \{U_\alpha\}$ of X, with connections ∇_α on each $E|U_\alpha$, then the Atiyah class is represented by the cocycle $\omega_{\alpha\beta} = \nabla_\beta - \nabla_\alpha$; this satisfies $\mathrm{d}\omega_{\alpha\beta} + \omega_{\alpha\beta}^2 = 0$ and $\mathrm{d}\,\mathrm{tr}\,\omega_{\alpha\beta} = 0$.

Lemma

If we have a Stein cover $\mathscr{U} = \{U_\alpha\}$ *of X, with connections* ∇_α *on each* $E|U_\alpha$*, then the Atiyah class is represented by the cocycle* $\omega_{\alpha\beta} = \nabla_\beta - \nabla_\alpha$ *; this satisfies* $\mathrm{d}\omega_{\alpha\beta} + \omega_{\alpha\beta}^2 = 0$ *and* $\mathrm{d}\,\mathrm{tr}\,\omega_{\alpha\beta} = 0$.

But normally Chern–Weil theory talks about characteristic classes as *curvatures* of connections, and here the curvature of $\mathrm{d} + \omega_{\alpha\beta}$ is *zero*.

Can we fix this?

The main idea

Our goal is to take all of the local data (i.e. the local resolutions by locally free sheaves, with local connections) and try to glue it together into some global object that "looks enough like" a connection in order to be able to apply Chern–Weil theory and recover the Chern classes as curvatures of these "almost connections".

The main idea

Our goal is to take all of the local data (i.e. the local resolutions by locally free sheaves, with local connections) and try to glue it together into some global object that "looks enough like" a connection in order to be able to apply Chern–Weil theory and recover the Chern classes as curvatures of these "almost connections".

Note that we can't hope to get an actual global resolution or an actual global connection, because we already know that, in most cases, these things don't exist.

The main idea

There are two things we need to glue:

1. the resolutions ; and
2. the connections.

We do both in slightly different ways, but always by using **simplices**. The former leads to the notion of **twisting cochains**; the latter to **admissible simplicial connections**.

The main idea

There are two things we need to glue:

1. the resolutions ; and
2. the connections.

We do both in slightly different ways, but always by using **simplices**. The former leads to the notion of **twisting cochains**; the latter to **admissible simplicial connections**.

The former is purely classical; the latter was studied by Green (1980) and O'Brian, Toledo, and Tong, but not in a categorical way.

Simplicial connections: locally free sheaves → characteristic classes

Double intersections

Given local connections ∇_α and ∇_β, the problem is that, over $U_{\alpha\beta}$, they there is no reason to expect them to agree.

Double intersections

Given local connections ∇_α and ∇_β, the problem is that, over $U_{\alpha\beta}$, they there is no reason to expect them to agree.

Rather than making a choice of which one to use over the intersection, we can actually just use **both**: consider the formal linear combination

$$t\nabla_\beta + (1-t)\nabla_\alpha$$

for $0 \leqslant t \leqslant 1$. This contains all the information of both connections (at $t=0$ and $t=1$).

Double intersections

Given local connections ∇_α and ∇_β, the problem is that, over $U_{\alpha\beta}$, they there is no reason to expect them to agree.

Rather than making a choice of which one to use over the intersection, we can actually just use **both**: consider the formal linear combination

$$t\nabla_\beta + (1-t)\nabla_\alpha$$

for $0 \leqslant t \leqslant 1$. This contains all the information of both connections (at $t = 0$ and $t = 1$).

This is now a "connection" on $U_{\alpha\beta} \times [0, 1]$.

Higher intersections

Given local connections $\nabla_{\alpha_0}, \ldots, \nabla_{\alpha_p}$, we can take their **barycentric average** in the same way:

$$\sum_{i=0}^{p} t_i \nabla_{\alpha_i}$$

where $\sum_{i=1}^{p} t_i \leqslant 1$ and $t_0 = 1 - \sum_{i=1}^{p} t_i$.

Higher intersections

Given local connections $\nabla_{\alpha_0}, \ldots, \nabla_{\alpha_p}$, we can take their **barycentric average** in the same way:

$$\sum_{i=0}^{p} t_i \nabla_{\alpha_i}$$

where $\sum_{i=1}^{p} t_i \leqslant 1$ and $t_0 = 1 - \sum_{i=1}^{p} t_i$.

This is now a “connection” on $U_{\alpha_0 \ldots \alpha_p} \times \Delta^p$.

Barycentric simplicial connection

We can thus construct the **barycentric simplicial connection** $\nabla^{\mu}_{\bullet}$, where, over each $U_{\alpha_0 \ldots \alpha_p}$,

$$\nabla^{\mu}_{p} = \sum_{i=0}^{p} t_i \nabla_{\alpha_i}.$$

Barycentric simplicial connection

We can thus construct the **barycentric simplicial connection** $\nabla^{\mu}_{\bullet}$, where, over each $U_{\alpha_0\ldots\alpha_p}$,

$$\nabla^{\mu}_{p} = \sum_{i=0}^{p} t_i \nabla_{\alpha_i}.$$

A very useful (but seemingly trivial) fact is that

$$\nabla^{\mu}_{p} = \nabla_{\alpha_0} + \sum_{i=1}^{p} t_i (\nabla_{\alpha_i} - \nabla_{\alpha_0}).$$

Barycentric simplicial connection

We can thus construct the **barycentric simplicial connection** $\nabla^\mu_\bullet$, where, over each $U_{\alpha_0\ldots\alpha_p}$,

$$\nabla^\mu_p = \sum_{i=0}^{p} t_i \nabla_{\alpha_i}.$$

A very useful (but seemingly trivial) fact is that

$$\nabla^\mu_p = \nabla_{\alpha_0} + \sum_{i=1}^{p} t_i (\nabla_{\alpha_i} - \nabla_{\alpha_0}).$$

So, if we define $\omega_{\alpha_0\alpha_i} = \nabla_{\alpha_i} - \nabla_{\alpha_0}$, then

$$\nabla^\mu_p \approx \mathrm{d} + \sum_{i=1}^{p} \omega_{\alpha_0\alpha_i}.$$

We can thus construct the **barycentric simplicial connection** $\nabla^{\mu}_{\bullet}$, where, over each $U_{\alpha_0\ldots\alpha_p}$,

$$\nabla^{\mu}_{p} = \sum_{i=0}^{p} t_i \nabla_{\alpha_i}.$$

A very useful (but seemingly trivial) fact is that

$$\nabla^{\mu}_{p} = \nabla_{\alpha_0} + \sum_{i=1}^{p} t_i(\nabla_{\alpha_i} - \nabla_{\alpha_0}).$$

So, if we define $\omega_{\alpha_0\alpha_i} = \nabla_{\alpha_i} - \nabla_{\alpha_0}$, then

$$\nabla^{\mu}_{p} \approx \mathrm{d} + \sum_{i=1}^{p} \omega_{\alpha_0\alpha_i}.$$

This brings the Atiyah class $\omega_{\alpha\beta}$ into play.

Barycentric simplicial connection

Writing $\overline{\omega}_p = \sum_{i=1}^{p} \omega_{\alpha_0 \alpha_i}$, we can show that

$$\kappa(\nabla_p^\mu) = \mathrm{d}\overline{\omega}_p + \overline{\omega}_p^2.$$

Barycentric simplicial connection

Writing $\overline{\omega}_p = \sum_{i=1}^{p} \omega_{\alpha_0 \alpha_i}$, we can show that

$$\kappa(\nabla_p^\mu) = \mathrm{d}\overline{\omega}_p + \overline{\omega}_p^2.$$

So this thing really looks like a connection with a curvature, and it encodes the Atiyah class, so we should be able to take its trace to get a characteristic class (Chern–Weil)!

Barycentric simplicial connection

Writing $\overline{\omega}_p = \sum_{i=1}^{p} \omega_{\alpha_0\alpha_i}$, we can show that

$$\kappa(\nabla_p^\mu) = \mathrm{d}\overline{\omega}_p + \overline{\omega}_p^2.$$

So this thing really looks like a connection with a curvature, and it encodes the Atiyah class, so we should be able to take its trace to get a characteristic class (Chern–Weil)!

N.B. $\mathrm{d}\omega_{\alpha_0\alpha_i} + \omega_{\alpha_0\alpha_i}^2 = 0$, but $\mathrm{d}\overline{\omega}_p + \overline{\omega}_p^2 \neq 0$. But this is what we wanted: we now have a "connection" which is **not** flat.

Formal properties

The $\overline{\omega}_p$ satisfy some properties that make it a **simplicial differential form**, i.e. a form on the product of the Čech nerve with the simplex category ($X^{\mathscr{U}}_{\bullet} \times \Delta^{\bullet}$) satisfying some conditions that ensures it descends to the fat geometric realisation.

The $\overline{\omega}_p$ satisfy some properties that make it a **simplicial differential form**, i.e. a form on the product of the Čech nerve with the simplex category ($X^{\mathscr{U}}_\bullet \times \Delta^\bullet$) satisfying some conditions that ensures it descends to the fat geometric realisation.

We won't talk about these here, but they are **very** important if we wish to give a formal definition of what a "simplicial connection" should be.

The problem now is that $\kappa(\nabla_\bullet^\mu)$ has extra terms in it that we don't want, i.e.

$$\kappa(\nabla_\bullet^\mu) = \left\{ -\sum_{i=1}^{p} \omega_{\alpha_0\alpha_i} \otimes \mathrm{d}t_i - \sum_{i=1}^{p} t_i \omega_{\alpha_0\alpha_i}^2 + \sum_{i,j=1}^{p} t_j t_i \omega_{\alpha_0\alpha_j} \omega_{\alpha_0\alpha_i} \right\}_{p\in\mathbb{N}}.$$

Fibre integration

The problem now is that $\kappa(\nabla^\mu_\bullet)$ has extra terms in it that we don't want, i.e.

$$\kappa(\nabla^\mu_\bullet) = \left\{ -\sum_{i=1}^{p} \omega_{\alpha_0\alpha_i} \otimes \mathrm{d}t_i - \sum_{i=1}^{p} t_i \omega^2_{\alpha_0\alpha_i} + \sum_{i,j=1}^{p} t_j t_i \omega_{\alpha_0\alpha_j}\omega_{\alpha_0\alpha_i} \right\}_{p\in\mathbb{N}}.$$

We can get rid of these in a "good" way (i.e. via a quasi-isomorphism from the simplicial de Rham complex to the usual de Rham complex) using Dupont's **fibre integration**, which simply integrates the $\kappa(\nabla^\mu_p)$ term over the geometric p-simplex.

Simplicial resolutions: coherent sheaves → locally free sheaves

Twisting cochains as ‘gluing up to homotopy’

Lemma

Coherent sheaves on paracompact complex-analytic manifolds can always be resolved by a ***twisting cochain****.*

Twisting cochains as ‘gluing up to homotopy’

Lemma

Coherent sheaves on paracompact complex-analytic manifolds can always be resolved by a ***twisting cochain****.*

A twisting cochain consists of “infinite homotopy gluing data”, i.e. it is the data of

- local resolutions by locally free sheaves
- quasi-inverse morphisms between these complexes
- homotopies witnessing these quasi-isomorphisms
- homotopies witness the failure of the above homotopies to commute with the quasi-isomorphisms
- homotopies between these homotopies witnessing …

Twisting cochains, formally

Twisting cochains are exactly Maurer–Cartan elements in some cochain complex:

$$\mathfrak{a} = \sum_{k\in\mathbb{N}} \mathfrak{a}^{k,1-k} \in \mathrm{Tot}^1\hat{\mathscr{C}}^\bullet(\mathscr{U}, \mathrm{End}^\star(V))$$

such that

$$\hat{\delta}\mathfrak{a} + \mathfrak{a}^2 = 0$$

(and also such that $\mathfrak{a}^{1,0}_{\alpha\alpha} = \mathrm{id}$ and $\mathfrak{a}^{1,0}_{\alpha} = \mathrm{d}_\alpha$).

Here V is the collection of local resolutions by locally free sheaves, and $\hat{\mathscr{C}}$ is the **deleted Čech complex**.

Green's resolution

Theorem (Green, 1980)

Let $\mathscr{E}$ be a coherent sheaf on a paracompact complex-analytic manifold $(X, \mathcal{O}_X)$ with "nice" Stein cover $\mathscr{U} = \{U_\alpha\}$. Then we can inductively modify a twisting cochain for $\mathscr{E}$ to obtain a "very nice" complex of locally free sheaves on the nerve that resolves $\mathscr{E}$.

Theorem (Green, 1980)

Let $\mathscr{E}$ be a coherent sheaf on a paracompact complex-analytic manifold $(X, \mathcal{O}_X)$ with "nice" Stein cover $\mathscr{U} = \{U_\alpha\}$. Then we can inductively modify a twisting cochain for $\mathscr{E}$ to obtain a "very nice" complex of locally free sheaves on the nerve that resolves $\mathscr{E}$.

In the same way that "simplicial connections" are somehow local connection all bundled up together, "locally free sheaves on the nerve" are somehow local locally free sheaves all bundled up together.

Vector bundles on the nerve vs. simplicial vector bundles

We refrain from calling these things "simplicial vector bundles" because that can be a misleading name: they are **not** simplicial objects in some category of vector bundles!

Vector bundles on the nerve vs. simplicial vector bundles

We refrain from calling these things "simplicial vector bundles" because that can be a misleading name: they are **not** simplicial objects in some category of vector bundles!

They are really lax limit objects, or cosimplicial diagrams, or decent data, or …

Vector bundles on the nerve vs. simplicial vector bundles

We refrain from calling these things “simplicial vector bundles” because that can be a misleading name: they are **not** simplicial objects in some category of vector bundles!

They are really lax limit objects, or cosimplicial diagrams, or decent data, or …

Definition (Vague)

A **sheaf on a simplicial space** $Y_\bullet$ is a collection of sheaves $\mathscr{F}^p$ on Y_p along with functorial morphisms

$$\mathscr{F}^\bullet(\alpha)\colon (Y_\bullet\alpha)^*\mathscr{F}^p \to \mathscr{F}^q$$

for all morphisms $\alpha\colon [p] \to [q]$ in Δ.

The Čech nerve is the simplicial space given by “blowing up” a cover.

Some other views of twisting cochains

There are lots!

- Twisted complexes
- Maurer–Cartan elements (i.e. deformations)
- dg-nerve
- bar/cobar stuff

Formal equivalences

The main theorem

Theorem

There is an equivalence of $(\infty, 1)$*-categories:*

$$\mathsf{hocolim}_{\mathscr{U}}\, \mathrm{LGreen}_{\nabla,0}(X^{\mathscr{U}}_{\bullet}) \simeq \mathrm{LCCoh}(X).$$

What next?

Generalisations

- Deligne cohomology

Generalisations

- Deligne cohomology
- Rigid analytic geometry, Artin stacks
- Equivariant cohomology
- Foliations
- etc.

Open questions

- Coherent complexes vs. complexes with coherent cohomology
- General approach for local-to-global constructions