Chain complexes and their morphisms

Let $(C_\bullet, d^C_\bullet)$ and $(D_\bullet, d^D_\bullet)$ be chain complexes of abelian groups. We are interested in what the "good" definition of morphisms $C_\bullet \to D_\bullet$ should be.

Let's start with the weakest definition:

Definition 1. A morphism $f_\bullet : C_\bullet \to D_\bullet$ is the data of group homomorphisms

$$f_i : C_i \to D_{i+m}$$

for all $i \in \mathbb{Z}$, where m is fixed (and called the degree of $f_\bullet$).

The problem with this definition is that it really only sees the graded structure of $C_\bullet$ and $D_\bullet$ and doesn't take their differentials into consideration.

But rather than trying to modify the definition to fix this problem, let's try something else...

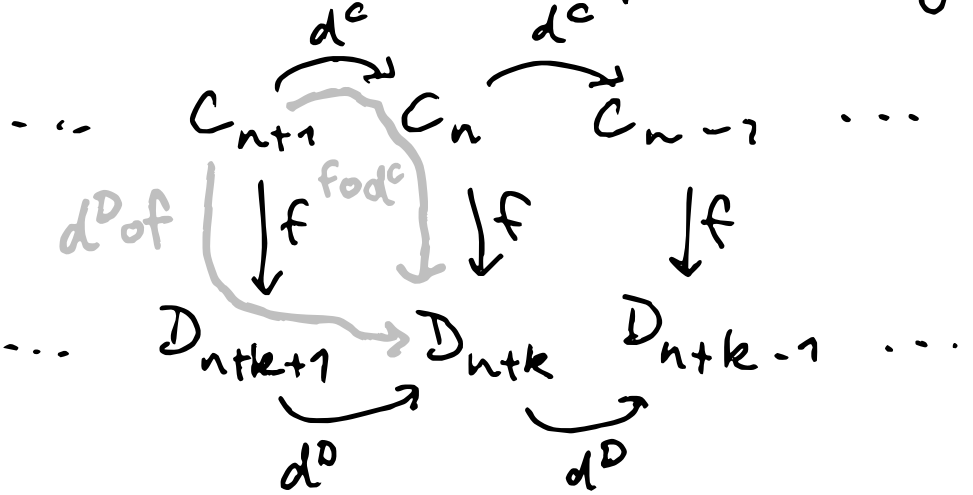
Define

$$\mathrm{Hom}^k(C_\bullet, D_\bullet) = \left\{ \text{morphisms } C_\bullet \to D_\bullet \text{ of degree } k \right\}.$$

What can we do with the differential $d^C_\bullet$?

Well note that we can <u>precompose</u> any $f \in \mathrm{Hom}^k(C_\bullet, D_\bullet)$ with $d^C_\bullet$ and obtain a morphism of degree $k+1$.

Similarly, we can <u>postcompose</u> any $f \in \mathrm{Hom}^k(C_\bullet, D_\bullet)$ with $d^D_\bullet$ and obtain another morphism of degree $k+1$.



So if we look at $f \circ d^C + d^D \circ f$ then we might be able to recover some information about the full (i.e. chain complex) structure of $C_\bullet$ and $D_\bullet$.

For this, we should expect to use the fact that $(d^C)^2 = 0$ and $(d^D)^2 = 0$. But note that

$$(f \circ d^C + d^D \circ f) \circ d^C = f \circ \underbrace{d^C \circ d^C}_{=0} + d^D \circ f \circ d^C$$

$$= d^D \circ f \circ d^C$$

and

$$d^D \circ (f \circ d^C + d^D \circ f) = d^D \circ f \circ d^C + \underbrace{d^D \circ d^D}_{=0} \circ f$$

$$= d^D \circ f \circ d^C.$$

This means that

$$\partial : \mathrm{Hom}^k(C_\bullet, D_\bullet) \to \mathrm{Hom}^{k+1}(C_\bullet, D_\bullet)$$

$$f \longmapsto f \circ d^C + (-1)^{k+1} d^D \circ f$$

is such that $\partial^2 = 0$, i.e. we get a

chain complex $\left(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet), \partial\right)$.

We already know that we get interesting things by looking at the cycles and boundaries of a chain complex, so let's do that here!

(N.B. We'll have cocycles and coboundaries because ∂ increases the degree)

Definition 2. A morphism $f_\bullet : C_\bullet \to D_\bullet$ is a cocycle $f \in K^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$, i.e.

$$f \in \mathrm{Hom}^0(C_\bullet, D_\bullet) \text{ such that}$$

$$\partial f = 0 \in \mathrm{Hom}^2(C_\bullet, D_\bullet)$$

Definition 3. A morphism $f_\bullet : C_\bullet \to D_\bullet$ is a representative of a cohomology class $[f] \in H^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$, i.e.

$$f \in \mathrm{Hom}^0(C_\bullet, D_\bullet) \text{ such that}$$

$$\partial f = 0 \in \mathrm{Hom}^1(C_\bullet, D_\bullet)$$

modulo $f \sim f'$ if $f = f' + \partial h$

for some $h \in \mathrm{Hom}^{-1}(C_\bullet, D_\bullet)$

So now we have three possible definitions of morphisms, but it turns out that we've seen them all before!

Claim. A morphism in the sense of Definition 2 is exactly a chain map, and a morphism in the sense of Definition 3 is exactly a chain homotopy class of chain maps

Proof. "Commuting with the differentials" is exactly saying that $\partial f = 0$. Saying that $f - f' = h \circ d^C + d^D \circ h$ is exactly saying that $f = f' + \partial h$ for $h \in \mathrm{Hom}^{-1}(C_\bullet, D_\bullet)$. □

In summary,

graded group homomorphisms $\longleftrightarrow \mathrm{Hom}^0(C_\bullet, D_\bullet)$

chain maps $\longleftrightarrow K^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$

chain homotopy classes of chain maps $\longleftrightarrow H^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$.

We actually have one last notion of morphism, which is a quasi-isomorphism, which we can define as an element of a localisation of $H^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$, but to fully explain this story we need to understand a bit more category theory / homological algebra than what is found in this course.

Let's now look at some specific examples! In particular, we'll try to understand the relation between quasi-isomorphisms and chain homotopy equivalences.

↳ i.e. understanding the relation between $H^0(\mathrm{Hom}(C,D))$ and $\mathrm{Hom}(H_\bullet(C), H_\bullet(D))$

Recall:

- A chain homotopy equivalence is a chain map $f: C_\bullet \to D_\bullet$ such that there exists a chain map $g: D_\bullet \to C_\bullet$ with $[gf] = [\mathrm{id}_{C_\bullet}] \in H^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$ and $[fg] = [\mathrm{id}_{D_\bullet}] \in H^0(\mathrm{Hom}^\bullet(D_\bullet, C_\bullet))$.
- A quasi-isomorphism is a chain map $f: C_\bullet \to D_\bullet$ such that the induced map $H_\bullet(f): H_\bullet(C_\bullet) \to H_\bullet(D_\bullet)$ is an isomorphism.

Firstly, note that chain homotopic maps induce identical morphisms in homology, i.e.

$$[f] = [g] \in H^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet))$$

$$\Rightarrow$$

$$H(f) = H(g) \in \mathrm{Hom}^0(H_\bullet(C_\bullet), H_\bullet(D_\bullet)).$$

We can write this rather succinctly (and abstractly!) as

$$H^0(\mathrm{Hom}^\bullet(C_\bullet, D_\bullet)) \subseteq \mathrm{Hom}^0(H_\bullet(C_\bullet), H_\bullet(D_\bullet))$$

which leads us to ask when the converse inclusion is also true (i.e. "when H commutes with Hom").

In general though, the converse does <u>not</u> hold. We can actually prove something slightly stronger: note that chain homotopy equivalences give quasi-isomorphisms; we will show that some quasi-isomorphisms do <u>not</u> come from chain homotopy equivalences.

Example. Let $C_\bullet = (0 \to 0 \to 0 \to \mathbb{Z}/2\mathbb{Z})$

and $D_\bullet = (0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z} \xrightarrow{\text{mod}2} \mathbb{Z}/2\mathbb{Z})$

and consider $f: C_\bullet \to D_\bullet$ given by

$$\begin{array}{ccccccc} 0 & \to & 0 & \to & 0 & \to & \mathbb{Z}/2\mathbb{Z} \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow \text{id} \\ 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} & \xrightarrow{\text{mod}2} & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

Then $H(f): H(C_\bullet) \to H(D_\bullet)$ is given by

$$\begin{array}{ccccccc} 0 & \to & 0 & \to & 0 & \to & \mathbb{Z}/2\mathbb{Z} \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow \text{id} \\ 0 & \to & 0 & \to & 0 & \to & \mathbb{Z}/2\mathbb{Z} \end{array}$$

which is a quasi-isomorphism.

Now, we can show that f does not admit a chain homotopy inverse by using the fact that the candidate for the inverse would be 0 (since we would have to have commutativity of $\begin{array}{ccc} 0 & \to & \mathbb{Z}/2\mathbb{Z} \\ \uparrow & & \uparrow \\ \mathbb{Z} & \to & \mathbb{Z}/2\mathbb{Z} \end{array}$).

But does there exist <u>any</u> quasi-isomorphism $C_\bullet \to D_\bullet$ that is also a chain homotopy equivalence?

No! And in fact the situation is even worse, since we can't even find a quasi-inverse for f, for exactly the same reason.

However, we can find a complex $P_\bullet$ along with a chain map $P_\bullet \to C_\bullet$ such that

(1) $P_\bullet \xrightarrow{\sim} C_\bullet$ is a quasi-isomorphism, and

(2) the composite $P_\bullet \xrightarrow{\sim} C_\bullet \to D_\bullet$ is a chain homotopy equivalence.

We can realise this by taking

$$P_\bullet = 0 \to 0 \to \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$$

along with $\varphi : P_\bullet \to C_\bullet$ given by

$$\begin{array}{ccccccc} 0 & \to & 0 & \to & \mathbb{Z} & \xrightarrow{\cdot 2} & \mathbb{Z} \\ \downarrow 0 & & \downarrow 0 & & \downarrow 0 & & \downarrow \text{mod } 2 \\ 0 & \to & 0 & \to & 0 & \to & \mathbb{Z}/2\mathbb{Z}. \end{array}$$

For those who have done some homological algebra or category theory, you might recognise that $P_\bullet \xrightarrow{\sim} C_\bullet$ is a projective resolution.

In fact, this hints at the bigger picture...

Theorem. (OFF SYLLABUS!!)

If we work with chain complexes of either

(i) modules over a field (recall that abelian groups are exactly modules over $\mathbb{Z}$)

or

(ii) projective modules over a (commutative) ring

then every quasi-isomorphism is also a chain homotopy equivalence.

"**Proof.**" When working over a field, every chain complex is formal (i.e. quasi-isomorphic to its homology, where $H_\bullet(C_\bullet)$ has the induced differential, which is zero, by definition). This follows from the fact that every complex here is split.

Then we can use the composition

$$C_\bullet \xrightarrow{\text{c.h.}} H_\bullet(C_\bullet) \xrightarrow{\sim} H_\bullet(D_\bullet) \xrightarrow{\text{c.h.}} D_\bullet \qquad \square$$

The whole story (dg-categories, A_∞-categories, model categories, ∞-categories) forms a part of categorical homotopy theory, and is beautiful, but also really big!