Graded rings, and the interaction between products and cohomology

The main object that we are interested in generalising is the *polynomial ring*

$$R[x] = \left\{ \begin{array}{c} \text{polynomials with coefficients} \\ \text{in } R \text{ and in the} \\ \text{variable } x \end{array} \right\}.$$

Note that polynomials come with a notion of *degree*:

$$\deg(r_m x^m + r_{m-1} x^{m-1} + \dots + r_0) = m$$

and that this is "respected" by addition and multiplication in the ring:

$$\deg(f+g) = \deg(f) = \deg(g)$$

$$\deg(fg) = \deg(f) + \deg(g).$$

This leads to the definition of *graded rings*.

- A <u>graded ring</u> is a ring R along with abelian groups $R_i \subset R$ (for $i \in \mathbb{N}$) such that

  (i) $R \cong \bigoplus_{i=0}^{\infty} R_i$

  (ii) $R_i R_j \subseteq R_{i+j}$.

- If $r \in R \setminus \{0\}$ is an element of R_i for some $i \in \mathbb{N}$, then we say that it is <u>homogeneous (of degree i)</u>. We write $|r|$ to denote the degree of a homogeneous element $r \in R$.

- We say that R is <u>graded commutative</u> if

$$r \cdot s = (-1)^{|r||s|} s \cdot r$$

for all homogeneous $r, s \in R$.

N.B. In general, for a graded ring R, being graded commutative is <u>not</u> the same as being commutative!

Example. $R[x] \cong \bigoplus_{i=0}^{\infty} R\langle x^i \rangle$

(the abelian group generated by monomials of degree i)

Note that $R[x]$ is **always** commutative (as long as R is...) but **not** always graded commutative, since

$$\underbrace{x}_{\text{degree } 1} \cdot \underbrace{x}_{\text{degree } 1} = (-1)\, x \cdot x$$

$$\Longleftrightarrow$$

$$2x^2 = 0$$

So we only have a graded commutative graded ring if $2 = 0$ in R (e.g. $\mathbb{Z}/2\mathbb{Z}$), **or** if we instead take

$$R[x]/(x^2) \quad \text{(the ring of dual numbers)}$$

OR if we define $\deg(x) = 2m$, (for some $m \in \mathbb{N}$) since then $(-1)^{|r||s|}$ is always equal to $+1$, and so graded commutativity is equivalent to commutativity.

Another important example is the exterior algebra:

$$\Lambda_R[\alpha] := \underbrace{R\langle 1\rangle}_{\text{the degree zero part is just } R} \oplus \underbrace{R\langle\alpha\rangle/(\alpha^2)}_{\text{the degree } |\alpha| \text{ part consists of monomials } s\alpha \text{ for } s\in R}$$

implicitly define $|\alpha| = 1$

i.e. elements of $\Lambda_R[\alpha]$ are of the form $r + s\alpha$, and we have that

$$(r+s\alpha)\cdot(r'+s'\alpha) = rr' + (rs'+r's)\,\alpha$$

since $\alpha^2 = 0$.

This is clearly commutative, but is it <u>graded</u> commutative? Yes!

$$\alpha\cdot\alpha = (-1)\,\alpha\cdot\alpha$$

since both sides are zero!

Note, however, that it is <u>not</u> (<u>in general</u>) free, since we <u>impose</u> the relation $\alpha^2 = 0$.

But if $R = \mathbb{Q}$ (say), then it **is** free as a **graded** commutative ring:

graded commutative $\Rightarrow 2\alpha^2 = 0$
$R = \mathbb{Q} \Rightarrow$ we can multiply by $1/2$
$\Rightarrow \alpha^2 = 0$

i.e. the fact that $\alpha^2 = 0$ is automatically implied by graded commutativity, and so $\Lambda_{\mathbb{Q}}[\alpha]$ **is** free.

In fact, for the same reason,

$\Lambda_R[\alpha]$ is free (as a **graded**-commutative ring) $\Longleftrightarrow$ $2 = 0$ **or** 2 is invertible in R

again: ⚠ graded commutative $\neq$ graded and commutative

Now, given graded rings R, S, we can give their tensor product a "natural" grading:

$$\deg(r \otimes s) := r + s$$

(and the multiplication has a minus sign:

$$(r \otimes s) \cdot (r' \otimes s') = (-1)^{|s||r'|} (rr') \otimes (ss')$$

defined for homogeneous r' and s, and then extended by linearity)

But what does all this have to do with algebraic topology?

Well, the cohomology of a space has a natural graded ring structure: multiplication is given by the cup product; grading is given by cohomological degree.

Fact. If Y is such that $H^\bullet(Y)$ is free and finitely generated then

$$H^\bullet(X \times Y) \cong H^\bullet(X) \otimes H^\bullet(Y)$$

isomorphic as graded rings!

Example. $H^n(S^1) = \begin{cases} \mathbb{Z} & n = 0,1 \\ 0 & \text{otherwise} \end{cases}$

If we write α to denote the generator of $H^1(S^1)$, then $\alpha^2 = \alpha \smile \alpha \in H^2(S^1)$ must be 0, since $H^2(S^1) = 0$. Thus

$$H^\bullet(S^1) \cong \mathbb{Z}\langle 1 \rangle \oplus \mathbb{Z}\langle \alpha \rangle / (\alpha^2)$$
$$=: \Lambda_{\mathbb{Z}}[\alpha]$$

Then

$$H^\bullet(S^1 \times S^1) \cong H^\bullet(S^1) \otimes H^\bullet(S^1)$$
$$\cong \Lambda_{\mathbb{Z}}[\alpha_1, \alpha_2]$$

(N.B. $\alpha_1 \alpha_2 = (-1)^{|\alpha_1||\alpha_2|} \alpha_2 \alpha_1 = -\alpha_2 \alpha_1$)

Example. $H^n(S^2) = \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1,\ n\geq 3 \\ \mathbb{Z} & n=2 \end{cases}$

If we write β to denote the generator of $H^2(S^2)$, then $\beta^2 = \beta \smile \beta \in H^4(S^2)$ must be 0, since $H^4(S^2)=0$. Thus

$$H^\bullet(S^2) \cong \mathbb{Z}\langle 1\rangle \oplus \mathbb{Z}\langle\beta\rangle \big/ (\beta^2) =: \Lambda_{\mathbb{Z}}[\beta]$$

but where $|\beta| := 2$. Recall the things we said about graded commutativity in the case of only having even degrees — this shows up when we look at the product:

$$H^\bullet(S^2\times S^2) \cong \Lambda_{\mathbb{Z}}[\beta_1,\beta_2]$$

but
$$\beta_1\beta_2 = (-1)^{|\beta_1||\beta_2|}\beta_2\beta_1$$
$$= \beta_2\beta_1$$

i.e. $\Lambda_{\mathbb{Z}}[\beta_1,\beta_2]$ is commutative

i.e. $\Lambda_{\mathbb{Z}}[\beta_1,\beta_2] \cong \mathbb{Z}[\beta_1,\beta_2] \big/ (\beta_1^2,\beta_2^2)$

In particular, $H^\bullet(S^1\times S^1) \not\cong H^\bullet(S^2\times S^2)$ as ungraded rings, since the former is not commutative!

What happens if $H^\bullet(Y)$ is finitely generated but <u>not</u> free? This means that it has <u>torsion</u> (by the classification of f.g. abelian groups), and the <u>Künneth theorem</u> tells us that this torsion is exactly the "correction term" needed:

<u>Corollary to the Künneth theorem.</u> (note that this is zero if $H^\bullet(Y)$ is free! (or actually if we work over a field instead of $\mathbb{Z}$ too))

$$H^\bullet(X \times Y) \cong \left(H^\bullet(X) \otimes H^\bullet(Y)\right) \oplus \overbrace{\mathrm{Tor}_{\bullet+1}\left(H^\bullet(X), H^\bullet(Y)\right)}$$

i.e.

$$H^n(X \times Y) \cong \left(\bigoplus_{i+j=n} H^i(X) \otimes H^j(Y)\right) \oplus \left(\bigoplus_{i+j=n+1} \mathrm{Tor}_1\left(H^i(X), H^j(Y)\right)\right)$$

<u>Example</u>. $H^\bullet(\mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1, n \geq 3 \\ \mathbb{Z}/2\mathbb{Z} & n=2 \end{cases}$

and $\mathrm{Tor}(-, \mathbb{Z}) = \mathrm{Tor}(\mathbb{Z}, -) = 0$

(since $\mathbb{Z}$ is free, and free ($\Rightarrow$ projective) $\Rightarrow$ flat)

and $\mathrm{Tor}(\mathbb{Z}/2\mathbb{Z}, \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$

(since $\mathrm{Tor}(A, \mathbb{Z}/m\mathbb{Z}) \cong \{a \in A \mid ma = 0\}$)

Thus

$$H^n(\mathbb{RP}^2 \times \mathbb{RP}^2) \cong \begin{cases} \mathbb{Z} & n=0 \\ 0 & n=1,\ n\geq 5 \\ (\mathbb{Z}/2\mathbb{Z})^2 & n=2 \\ \mathbb{Z}/2\mathbb{Z} & n=3 \\ \mathbb{Z}/2\mathbb{Z} & n=4 \end{cases}$$

This is the "correction" term given by Tor!

Using either the UCT or the LES associated to the SES $0 \to \mathbb{Z} \hookrightarrow \mathbb{Q} \twoheadrightarrow \mathbb{Q}/\mathbb{Z}$, we can also calculate

$$H^n(\mathbb{RP}^2 \times \mathbb{RP}^2; \mathbb{Q}/\mathbb{Z})$$

why is this choice of coefficient interesting? because it's "good at seeing torsion", i.e.

$$\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) = \mathrm{tors}(A)$$

(try it!)

Some useful cohomology rings

we can't use $\mathbb{Z}$ coefficients since then it wouldn't be graded commutative

$$H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]/(\alpha^{n+1})$$

$|\alpha|=1$

$$H^\bullet(\mathbb{RP}^\infty;\mathbb{Z}/2\mathbb{Z}) \cong (\mathbb{Z}/2\mathbb{Z})[\alpha]$$

$|\alpha|=1$

this looks like "$\lim_{n\to\infty} H^\bullet(\mathbb{RP}^n;\mathbb{Z}/2\mathbb{Z})$" ...how interesting...

We can use $\mathbb{Z}$ coefficients here since $|\beta|$ is even

$$H^\bullet(\mathbb{CP}^n) \cong \mathbb{Z}[\beta]/(\beta^{n+1})$$

$|\beta|=2$

$$H^\bullet(\mathbb{CP}^\infty) \cong \mathbb{Z}[\beta]$$

$|\beta|=2$

if we really want to use $\mathbb{Z}$ coefficients then we still get a graded ring, but we just can't write it in one simple form for all n...

probably not worth memorising these...

$$H^\bullet(\mathbb{RP}^{2n}) \cong \mathbb{Z}[\alpha]/(2\alpha, \alpha^{n+1})$$

$|\alpha|=2$

not $2n+1$

$$H^\bullet(\mathbb{RP}^{2n+1}) \cong \mathbb{Z}[\alpha,\varepsilon]/(2\alpha, \alpha^{n+1}, \varepsilon^2, \alpha\varepsilon)$$

$|\alpha|=2$

$|\varepsilon|=2n+1$
(ε is a generator of H^{2n+1})