Left adjoint to Yoneda

$$Y : \mathcal{C} \to Psh(\mathcal{C})$$
$$c \mapsto Hom(-, c)$$

we want $Y^L : Psh(\mathcal{C}) \longrightarrow \mathcal{C}$

Let $F \in Psh(\mathcal{C})$, i.e. $F : \mathcal{C}^{op} \to Set$

Then $Y^L(F) \in \mathcal{C}$ should just be an object of $\mathcal{C}$.

Claim. $Y^L(F) =$ " $\underset{\substack{x \in \mathcal{C} \\ z \in F(x)}}{\mathrm{colim}}\ x$ " (when this colimit exists — i.e. if $\mathcal{C}$ is total)

But what does this colimit actually mean? It's very concisely written, with some abuses of notation!

Consider $\int F :=$ "the category of elements of F"

whose objects are pairs (x, z) with $x \in \mathcal{C}$ and $z \in F(x)$ and morphisms $(x,z) \to (x',z')$ are $x \xrightarrow{f} x'$ (in $\mathcal{C}$) such that $F(f)(z') = z$

Now look at the forgetful functor

$Forget : \int F \to \mathcal{C}$ given by $Forget((x,z)) = x$.

$$\boxed{\text{Then } Y^L(F) = \underset{\int F}{\mathrm{colim}}\ Forget}$$