Mapping toruses

Let X be a manifold, and pick some $f: X \to X$
We define the mapping torus of f to be the quotient

$$M_f := \frac{X \times [0,1]}{\langle (x,0) \sim (f(x),1) \rangle}$$

We have a surjection

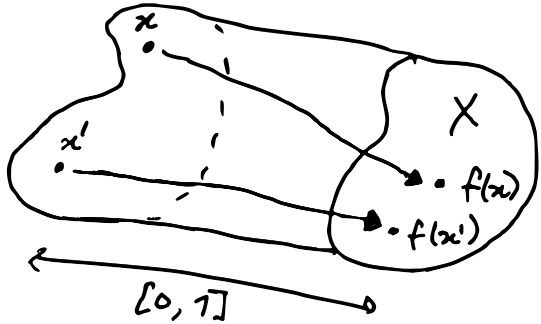
$$\pi : M_f \longrightarrow S^1$$
$$[(x,t)] \longmapsto t$$

we land in S^1, not $[0,1]$, because every $[(x,0)]$ is identified with some point $[(y,1)]$ (in particular for $y = f(x)$)

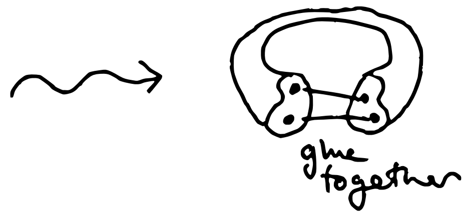
and the fibres of this surjection are homeomorphic to X, i.e.

$$\pi^{-1}(t) \cong X \quad \text{for all } t \in S^1.$$

(To use some technical language that is outside the syllabus, we have a fibre bundle $\pi : M_f \to S^1$ with fibre X.)



(think about what this looks like if f is constant)



Writing $\bar{X}$ to mean the image of $X \times \{0\}$ in M_f under the quotient map $X \times [0,1] \longrightarrow M_f$, we claim (without proof!) that

$$M_f / \bar{X} \cong (\Sigma X) \vee S^1$$

(unreduced) suspension

wedge sum

We also claim (without proof!) that $(M_f, \bar{X})$ is a <u>good pair</u>

For a specific example, let's take the <u>torus</u>

$$T = \mathbb{R}^2 / \mathbb{Z}^2$$

and let $f : T \longrightarrow T$ be given by some (2×2)-matrix of integers $m \in M_2(\mathbb{Z})$ acting on the plane $\mathbb{R}^2$. In fact, let's take

$$f = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad g = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

Calculating $H_3(M_f)$

Since the pair $(M_F, \bar{T})$ is good, we have the LES of a pair:

$$\begin{tikzcd}
\tilde{H}_\bullet(\bar{T}) \arrow[rr, "i_*"] & & \tilde{H}_\bullet(M_f) \arrow[dl, "j_*"] \\
& H_\bullet(M_f, \bar{T}) \arrow[ul, "\partial\,[-1]"] &
\end{tikzcd}$$

For various reasons, it turns out that H_3 of the mapping torus is usually the most interesting, so let's look at the relevant part of the above LES:

$$\underbrace{\tilde{H}_3(\bar{T})}_{\cong \tilde{H}_3(T) \cong 0} \xrightarrow{i_*} \underbrace{\tilde{H}_3(M_f)}_{\text{we want to calculate this ...}} \xrightarrow{j_*} \underbrace{H_3(M_f, \bar{T})}_{\text{... so we first need to calculate this!}} \xrightarrow{\partial} \underbrace{\tilde{H}_2(\bar{T})}_{\cong H_2(T) \cong \mathbb{Z}}$$

Useful corollary to the excision theorem.

If $A \subseteq B \subseteq X$ are such that

(i) $\bar{A} \subseteq \mathring{B}$
(ii) $A \hookrightarrow B$ is a deformation retract

then $H_\bullet(X, A) \xrightarrow{\cong} H_\bullet(X/A, *) \left(\cong \tilde{H}_\bullet(X/A)\right)$

This corollary tells us that

$$H_\bullet(M_f, \bar{T}) \cong H_\bullet(M_f/\bar{T})$$

(by taking a small open neighbourhood of $\bar{T}$ inside M_f)

Thus

$$\begin{aligned}
H_3(M_f, \bar{T}) &\cong H_3(M_f/\bar{T}) && \text{by excision} \\
&\cong H_3(\Sigma T \vee S^1) && \text{by the claim near the beginning} \\
&\cong H_3(\Sigma T) \vee H_3(S^1) && \text{by properties of the wedge sum} \\
&\cong H_3(\Sigma T) && \text{since } H_3(S^1) \cong 0 \\
&\cong H_2(T) && \text{by properties of suspension} \\
&\cong \mathbb{Z}
\end{aligned}$$

and so our exact sequence looks like

$$0 \longrightarrow H_3(M_f) \longrightarrow \mathbb{Z} \xrightarrow{\partial} \mathbb{Z}$$

but how does ∂ act?

We claim that the map

$$m_* : H_2(T) \longrightarrow H_2(T)$$

induced by $m \in M_2(\mathbb{Z})$ is given exactly by <u>multiplication</u> by <u>det(m)</u>.

(we also won't prove this, but note that it seems believable ...)

<u>Claim</u>: $\partial : H_3(M_f, \bar{T}) \to \tilde{H}_2(\bar{T})$ is given by

$$\mathbb{Z} \longrightarrow \mathbb{Z}$$
$$n \longmapsto (1-\det f)\, n$$

<u>Proof</u>: (we'll come back to this after finishing our calculation!)

So $$0 \longrightarrow H_3(M_f) \longrightarrow \mathbb{Z} \xrightarrow{\cdot 2} \mathbb{Z}$$

is exact (since $\det f = -1$), and so

$$H_3(M_f) \cong 0$$

But note that, if we do this calculation in $\mathbb{Z}/2\mathbb{Z}$ then $\partial = \cdot 2 = \cdot 0$, and so $H_3(M_f; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}$.

For g we apply the same argument to see that

$$H_3(M_g) \cong 0 \quad \text{and} \quad H_3(M_g; \mathbb{Z}/2\mathbb{Z}) = 0$$

since $\partial = 1 - \det g = 1$.

Clearly the really difficult part is understanding <u>why</u> $\partial = 1 - \det f$. For this, see

Hatcher, Example 2.48

but note that this is not particularly trivial (especially since Hatcher works with something <u>homeomorphic</u> to M_f, and not exactly with our definition of M_f).

The idea is a bit similar to Mayer-Vietoris in that some kernel is given by elements of the form $(\alpha, -\alpha)$, and we then apply $(id_* \times f_*)$ to this.