Revision sheets – Part C 2016

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Abstract

These are the revision notes that I made for my final exams in my masters course. They have never been updated since, so there are probably plenty of mistakes. They find themselves uploaded in the hope that some other poor student one day finds them even remotely useful.

Contents

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1 Fundamentals of category theory

We often abuse notation and write $x \in \mathcal{C}$ to mean $x \in \mathrm{ob}\mathcal{C}$.

1.1 Functors

When we say ‘functor’ we mean a *covariant* functor; we write a contravariant functor $F' : \mathcal{C} \to \mathcal{D}$ as the covariant functor $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

1.1.1 Types of functors

Let $F : \mathcal{C} \to \mathcal{D}$ be a functor.

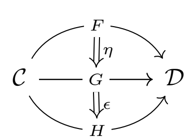
- By definition, F induces a map $\mathrm{Hom}_{\mathcal{C}}(x,y) \to \mathrm{Hom}_{\mathcal{D}}(F(x),F(y))$. We say that F is
  - **faithful** if this map is injective;
  - **full** if this map is surjective.
- The set $\{F(c) \mid c \in \mathcal{C}\}$ is called the **image** of F, and its closure under isomorphism $\{d \in \mathcal{D} \mid d \cong F(c) \text{ for some } c \in \mathcal{C}\}$ is called the **essential image** of F.
- We say that F is **essentially surjective** (or **dense**) if, for all $d \in \mathcal{D}$, there exists some $c \in \mathcal{C}$ such that $F(c) \cong d$, i.e. if the essential image of F is all of $\mathcal{D}$.
- Let $F : \mathcal{C}^{\mathrm{op}} \to \mathsf{Set}$. We say that F is **representable** if it is in the essential image of the **Yoneda functor** $Y_{(-)} : \mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{\mathrm{op}}, \mathsf{Set})$ given by $Y_x = \mathrm{Hom}_{\mathcal{C}}(-,x)$.
- Let $F : \mathcal{C} \to \mathsf{Set}$. We say that F is **corepresentable** if it is in the essential image of the **co-Yoneda functor** $Y^{(-)} : \mathcal{C}^{\mathrm{op}} \to \mathrm{Fun}(\mathcal{C}, \mathsf{Set})$ given by $Y^x = \mathrm{Hom}_{\mathcal{C}}(x,-)$.

1.1.2 Equivalence of categories

- Let $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ be functors. We say that F and G form an **equivalence of categories** if there exists natural *isomorphisms* $e : \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and $\epsilon : FG \Rightarrow \mathrm{id}_{\mathcal{D}}$.
- *A functor $F : \mathcal{C} \to \mathcal{D}$ is part of an equivalence of categories iff it is fully faithful and essentially surjective*

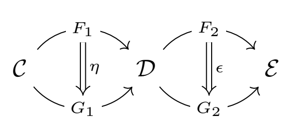
1.2 Natural transformations

- A **natural isomorphism** is a natural transformation whose constituent morphisms are all isomorphisms.
- We can compose natural transformations in three ways:
  - **Vertical composition:**



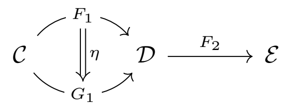
$$F \Rightarrow H$$
$$F(x) \xrightarrow{\eta_x} G(x) \xrightarrow{\epsilon_x} H(x)$$

  - **Horizontal composition:**



$$F_2F_1 \Rightarrow G_2G_1$$
$$F_2F_1(x) \xrightarrow{F_2(\eta_x)} F_2G_1(x) \xrightarrow{\epsilon_{G_1(x)}} G_2G_1(x)$$

– **Whiskering:** (really just horizontal composition with $G_2 = F_2$ and $\epsilon = \mathrm{id}_{G_2}$)



$$F_2F_1 \Rightarrow F_2G_1$$
$$F_2F_1(x) \xrightarrow{F_2(\eta_x)} F_2G_1(x)$$

1.3 Adjunctions

1.3.1 Equivalent definitions

Let $F\colon \mathcal{C} \to \mathcal{D}$ and $G\colon \mathcal{D} \to \mathcal{C}$ be functors. We say that F and G are **adjoint**, with F **left-adjoint** and G **right-adjoint**, and write $F \dashv G$, if any of the following conditions hold:

- **Units:** there exist natural transformations $e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$, the **unit**, and $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, the **counit**, such that the following two composites are both the identity natural transformation:
  1. $F \overset{\mathrm{id}_F \circ e}{\Rightarrow} FGF \overset{\epsilon \circ \mathrm{id}_F}{\Rightarrow} F$;
  2. $G \overset{e \circ \mathrm{id}_G}{\Rightarrow} GFG \overset{\mathrm{id}_G \circ \epsilon}{\Rightarrow} G$.
- **Natural isomorphisms:** there is an isomorphism $\mathrm{Hom}_{\mathcal{D}}(F(x), y) \cong \mathrm{Hom}_{\mathcal{C}}(x, G(y))$that is natural in both x and y.
- **Initial objects:** there exists a natural transformation $e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ such that $(F(x), e_x)$ is initial in $(x \Rightarrow G)$ for all $x \in \mathcal{C}$.
- **Representable functors:** the formal right-adjoint functor G^{formal} defined by

$$\begin{aligned} G^{\text{formal}}\colon \mathcal{D} &\to \mathrm{Fun}(\mathcal{C}^{\text{op}}, \mathsf{Set}) \\ y &\mapsto \mathrm{Hom}_{\mathcal{C}}(F(-), y) \end{aligned}$$

  is representable for all $y \in \mathcal{D}$.

Work through all of these definitions and show that they're equivalent.

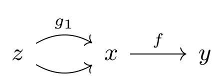
1.3.2 Properties

- *A functor $F\colon \mathcal{C} \to \mathcal{D}$ has a* left adjoint *iff $(y \Rightarrow F)$ has an initial object for every $y \in \mathcal{D}$; it has a* right adjoint *iff $(F \Rightarrow y)$ has a final object for every $y \in \mathcal{D}$.*
- The **solution set condition** on a locally small category $\mathcal{C}$ with small colimits is the following: there exists a small set I and a family of objects $\{c_i\}_{i \in I}$ of $\mathcal{C}$ such that, for every $x \in \mathcal{C}$, there exists $i \in I$ with $\mathrm{Hom}_{\mathcal{C}}(x, c_i)$ is non-empty.
- **Adjoint functor theorem:** *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between locally small categories, where $\mathcal{C}$ has small colimits. Then F has a* right adjoint *iff $(F \Rightarrow y)$ satisfies the solution set condition for all $y \in \mathcal{D}$ and F preserves colimits.*

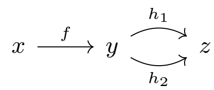
1.4 Monomorphisms and epimorphisms

Let $f\colon x \to y$ be some morphism in $\mathcal{C}$, and $z \in \mathcal{C}$ some object. Then f is

- a **monomorphism** if for all $g_1, g_2\colon z \to x$ such that $f \circ g_1 = f \circ g_2$ we have that $g_1 = g_2$



- an **epimorphism** if for all $h_1, h_2 \colon y \to z$ such that $h_1 \circ f = h_2 \circ f$ we have that $h_1 = h_2$



- a **regular monomorphism** if $f = \mathrm{eq}(y \rightrightarrows y')$ for some $y' \in \mathcal{C}$

$$x \xrightarrow{f} y \rightrightarrows y'$$

- a **regular epimorphism** if $f = \mathrm{coeq}(x' \rightrightarrows x)$ for some $x' \in \mathcal{C}$

$$x' \rightrightarrows x \xrightarrow{f} y$$

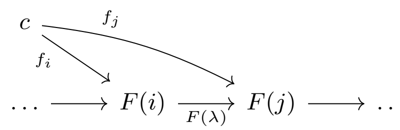
If f is a regular monomorphism (epimorphism) then it is a monomorphism (epimorphism).

1.5 (Co)Limits

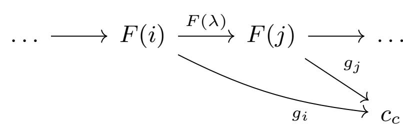
1.5.1 Definitions using diagrams

Let $F \colon I \to \mathcal{C}$ be a functor.

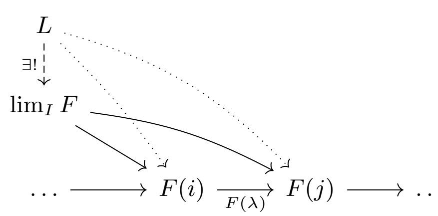
- We call $F \colon I \to \mathcal{C}$ a **diagram of shape** I.
- A **cone of** F is an object $c \in \mathcal{C}$ together with morphisms $f_i \colon c \to F(i)$ for all $i \in I$ such that, if $\lambda \colon i \to j$ then $F(\lambda) \circ f_i = f_j$.



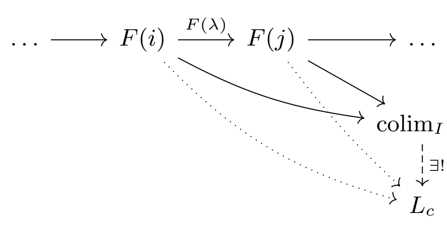
- A **cocone of** F is an object $c_c \in \mathcal{C}$ together with morphisms $g_i \colon F(i) \to c_c$ for all $i \in I$ such that, if $\lambda \colon i \to j$ then $g_j \circ F(\lambda) = g_i$.



- A **limit** of F is a final object in the category of cones of F. That is, it is a cone $\lim_I F \in \mathcal{C}$ of F, and it is universal with respect to this property: if $L \in \mathcal{C}$ is another cone of F then there exists a unique morphism $L \to \lim_I F$.



- A **colimit** of F is an initial object in the category of cocones of F. That is, it is a cocone $\mathrm{colim}_I F \in \mathcal{C}$ of F, and it is universal with respect to this property: if $L_c \in \mathcal{C}$ is another cocone of F then there exists a unique morphism $\mathrm{colim}_I F \to L_c$.



- *We have the relation* $\mathrm{colim}_I F = \lim_{I^{op}} F^{op}$*, where* $F^{op} \colon I^{op} \to \mathcal{C}^{op}$.

1.5.2 Definitions using the diagonal functor

Let $F\colon I \to \mathcal{C}$ be a functor.

- The **diagonal functor** is $\Delta_{(-)}\colon \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C})$, where $\Delta_x\colon i \mapsto x$ is the constant functor and $\Delta_f\colon \Delta_x \Rightarrow \Delta_y$ for $f\colon x \to y$ is given by $(\Delta_f)_c = f$ for all $c \in \mathcal{C}$.
- *A cone C on F is equivalent to a natural transformation $\Delta_C \Rightarrow F$.*

  This follows from simply unpacking all the definitions.
- *If $\mathcal{C}$ has all limits of shape I then* $\lim_I\colon \mathrm{Fun}(I,\mathcal{C}) \to \mathcal{C}$ *is* right adjoint *to the diagonal functor* $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C})$.

  Since any cone C on F is equivalent to a natural transformation $\Delta_C \Rightarrow F$, and any cone C on F also has a unique morphism $C \to \lim_I F$, we have a *natural* isomorphism $\mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta_C, F) \cong \mathrm{Hom}_{\mathcal{C}}(C, \lim_I F)$.
- Dually, *if $\mathcal{C}$ has all colimits of shape I then* $\mathrm{colim}_I\colon \mathrm{Fun}(I,\mathcal{C}) \to \mathcal{C}$ *is* left adjoint *to the diagonal functor* $\Delta\colon \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C})$.

$$\mathrm{colim}_I \dashv \Delta \dashv \lim_I$$

1.5.3 Existence

- A category $\mathcal{C}$ **has all limits of shape** I if $\lim_I F$ exists for all functors $F\colon I \to \mathcal{C}$.
- A category $\mathcal{C}$ **has limits** (or **is complete**) if it has limits of shape I for all *small* categories I.
- A category $\mathcal{C}$ **has finite limits** if it has limits of shape I for all finite categories I.

1. *If $\mathcal{C}$ has all limits of shape I and J then it has all limits of shape $I \times J$, and the following 'Fubini formula' holds:*

   $$\lim_{I\times J} F \cong \lim_I \lim_J F \cong \lim_J \lim_I F.$$

   Let $F\colon I \times J \to \mathcal{C}$ be a diagram, which we can think of as a functor $F\colon J \to \mathrm{Fun}(I,\mathcal{C})$. We have diagonal functors

   $$\begin{aligned}&\Delta^I_{(-)}\colon \mathcal{C} \to \mathrm{Fun}(I,\mathcal{C})\\ &\Delta^J_{(-)}\colon \mathrm{Fun}(I,\mathcal{C}) \to \mathrm{Fun}(J,\mathrm{Fun}(I,\mathcal{C}))\\ &\Delta^{I\times J}_{(-)}\colon \mathcal{C} \to \mathrm{Fun}(I\times J,\mathcal{C}).\end{aligned}$$

   Consider the following chain of natural isomorphisms (and similarly for $\lim_J \lim_I F$):

   $$\begin{aligned}\mathrm{Hom}_{\mathcal{C}}(x, \lim_I \lim_J F) &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta^I_x, \lim_J F)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(J,\mathrm{Fun}(I,\mathcal{C}))}(\Delta^J_{\Delta^I_x}, F)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(I\times J,\mathcal{C})}(\Delta^{I\times J}_x, F)\\ &\cong \mathrm{Hom}_{\mathcal{C}}(x, \lim_{I\times J} F).\end{aligned}$$

2. *The 'Fubini formula' also holds for colimits.*
3. *A category $\mathcal{C}$ has (co)limits iff*
   (a) *it has (co)products and (co)equalizers;*
   (b) *it has cofiltered (filtered) limits and finite (co)limits.*

   (a) Say that $\mathcal{C}$ has products and equalizers, and let $F\colon I \to \mathcal{C}$. For every $f\colon i \to j$ in I we define two morphisms $\left(\prod_{k\in I} F(k)\right) \to F(j)$. The first is simply the projection map from $k = j$; the second is the composition of the project map from $k = i$ followed by $F(f)$. Then both of these maps factor uniquely through $\prod_{(m\to n)\in \mathrm{Fun}([1],I)} F(n)$, where $\mathrm{Fun}([1],I)$ is the category of arrows in I, i.e. $m,n \in I$ together with some morphism $m \to n$. Then (checking universal properties) $\lim_I F = \mathrm{eq}\left(\prod_{k\in I} F(k) \rightrightarrows \prod_{(m\to n)\in \mathrm{Fun}([1],I)} F(n)\right)$.

(b) Say that $\mathcal{C}$ has cofiltered and finite limits. By the previous statement and the next statement, it is enough to show that $\mathcal{C}$ has products, since we get equalizers from finite limits. Let $F\colon I \to \mathcal{C}$ with I small. Define I^+ to be finite subsets of I, with morphisms given by inclusion. Then $(I^+)^{\text{op}}$ is cofiltered. Define $F^+\colon (I^+)^{\text{op}} \to \mathcal{C}$ by $J \mapsto \lim_J F_J$. Then (checking universal properties) $\lim_{(I^+)^{\text{op}}} F^+ = \prod_I F$.

4. *$\mathcal{C}$ has finite (co)limits iff it has binary (co)products, (co)equalizers, and a final (initial) object.*

Let $\mathcal{C}, \mathcal{D}$ be categories. Recall that the **presheaf category** $\text{PShv}(\mathcal{C})$ is defined to be $\text{Fun}(\mathcal{C}^{\text{op}}, \text{Set})$, and that the Yoneda embedding $\mathcal{C} \hookrightarrow \text{PShv}(\mathcal{C})$ given by $x \mapsto \text{Hom}_{\mathcal{C}}(-, x)$ is fully faithful.

1. *If $\mathcal{D}$ has limits of shape I then the composite*

$$\text{Fun}(I, \text{Fun}(\mathcal{C}, \mathcal{D})) \cong \text{Fun}(\mathcal{C}, \text{Fun}(I, \mathcal{D})) \to \text{Fun}(\mathcal{C}, \mathcal{D}).$$

is the limit functor, i.e. limits in $\text{Fun}(\mathcal{C}, \mathcal{D})$ are computed pointwise, and dually for colimits.

We claim that the adjunction $(\Delta \dashv \lim_I)$ between categories $\mathcal{D} \rightleftarrows \text{Fun}(I, \mathcal{D})$ induces an adjunction

$$\text{Fun}(\mathcal{C}, \mathcal{D}) \rightleftarrows \text{Fun}(\mathcal{C}, \text{Fun}(I, \mathcal{D})).$$

Then we claim that the composition

$$\text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(\mathcal{C}, \text{Fun}(I, \mathcal{D}) \cong \text{Fun}(I, \text{Fun}(\mathcal{C}, \mathcal{D}))$$

coincides with the constant functor

$$\text{Fun}(\mathcal{C}, \mathcal{D}) \to \text{Fun}(I, \text{Fun}(\mathcal{C}, \mathcal{D})),$$

where the first functor comes from the constant functor $\Delta\colon \mathcal{D} \to \text{Fun}(I, \mathcal{D})$.

2. *Every presheaf is a colimit of representable presheaves. In particular, for a presheaf F we have an isomorphism $F \cong \text{colim}_{(*\Rightarrow F)^{op}} P$, where $P(x, y) = Y_x$.*

We have $P\colon (* \Rightarrow F)^{\text{op}}$ defined by $P(x, y) = Y_x$ and the constant functor $\Delta\colon \text{PShv}(\mathcal{C}) \to \text{Fun}((* \Rightarrow F)^{\text{op}}, \text{PShv}(\mathcal{C}))$. If $\mathcal{C}$ is small then so too is $(* \Rightarrow F)^{\text{op}}$. We claim that

$$\text{Hom}_{\text{PShv}(\mathcal{C})}(F, G) \cong \text{Hom}_{\text{Fun}((*\Rightarrow F)^{\text{op}}, \text{PShv}(\mathcal{C}))}(P, \Delta_G).$$

Then the result follows from the fact that colimits are left adjoint to the constant functor.

3. *Let $\mathcal{C}$ be small and $\mathcal{D}$ have small colimits. Then there is an equivalence of categories*

$$\text{Fun}^{\text{colim}}(\text{PShv}(\mathcal{C}), \mathcal{D}) \cong \text{Fun}(\mathcal{C}, \mathcal{D}).$$

Work through the proof of this (p. 34 of lecture notes).

1.5.4 Inverse and direct limits

- A **poset** P is a set with a **partial order**, i.e. for all $a, b, c \in P$ the following hold:
  1. $a \leqslant a$
  2. $a \leqslant b \leqslant c \implies a \leqslant c$
  3. $a \leqslant b \leqslant a \implies a = b$
- A **directed poset** is a poset D such that every pair of elements has an upper bound, i.e. for all $x, y \in D$ there exists $u \in D$ such that $x \leqslant d$ and $y \leqslant d$.

Let D be a directed poset (usually $\mathbb{N}$).

- An **inverse** (or **projective**) **limit** is a limit of shape D^{op}, written as $\varprojlim_D$.
- A **direct** (or **inductive**) **limit** is a colimit of shape D, written as $\varinjlim_D$.

When $\mathcal{C} = \mathsf{Ab}$ (or, more generally, R-mod) and D is a directed poset, we have the following descriptions of projective and inductive limits:

$$\varprojlim_D F = \left\{ (x_\alpha) \in \prod_{\alpha \in D} F(\alpha) \;\middle|\; F(\alpha \leqslant \beta)(x_\beta) = x_\alpha \text{ for all } \alpha \leqslant \beta \right\}$$
$$\varinjlim_D F = \left(\coprod_{\alpha \in D} F(\alpha) \right) \Big/ \left\{ x_\alpha \sim F(\alpha \leqslant \beta)(x_\alpha) \text{ for } x_\alpha \in F(\alpha) \right\}$$

1.5.5 Examples

I	$\lim_I F$	$\operatorname{colim}_I F$
empty category	final object in $\mathcal{C}$	initial object in $\mathcal{C}$
discrete category	product indexed by I	coproduct indexed by I
$*$	$F(*)$	$F(*)$
$* \rightrightarrows *$	equaliser	coequaliser
$* \to * \leftarrow *$	pullback	–
$* \leftarrow * \to *$	–	pushout

Table 1: (Co)Limits of shape I for various categories I, where $*$ is an arbitrary single object.

1. See Table 1 for some general examples of limits of shape I for various categories I.
2. A pullback can be written as an equaliser and a product: $X \times_Z Y \cong \operatorname{eq}(X \times Y \rightrightarrows Z)$
3. When $\mathcal{C} = R$-mod and I is a directed poset (usually taken to be $\mathbb{N}$), we have the following identities:
   - $\lim(A \twoheadleftarrow A^2 \twoheadleftarrow A^3 \twoheadleftarrow \ldots) = \prod_{n=1}^\infty A$
   - $\operatorname{colim}(A \rightarrowtail A^2 \rightarrowtail A^3 \rightarrowtail \ldots) = \coprod_{n=1}^\infty A$
   - $\lim(\mathbb{Z}/p\mathbb{Z} \twoheadleftarrow \mathbb{Z}/p^2\mathbb{Z} \twoheadleftarrow \mathbb{Z}/p^3\mathbb{Z} \twoheadleftarrow \ldots) = \mathbb{Z}_p$ (the **p-adic integers**)
   - $\operatorname{colim}(\mathbb{Z}/p\mathbb{Z} \rightarrowtail \mathbb{Z}/p^2\mathbb{Z} \rightarrowtail \mathbb{Z}/p^3\mathbb{Z} \rightarrowtail \ldots) = \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$
   - $\lim(\mathbb{Z}/1!\mathbb{Z} \twoheadleftarrow \mathbb{Z}/2!\mathbb{Z} \twoheadleftarrow \mathbb{Z}/3!\mathbb{Z} \twoheadleftarrow \ldots) = \widehat{\mathbb{Z}}$ (the **profinite completion of $\mathbb{Z}$**)
   - $\operatorname{colim}(\mathbb{Z}/1!\mathbb{Z} \rightarrowtail \mathbb{Z}/2!\mathbb{Z} \rightarrowtail \mathbb{Z}/3!\mathbb{Z} \rightarrowtail \ldots) = \mathbb{Q}/\mathbb{Z}$

   We also have the following isomorphisms:
   - $\mathbb{Q}/\mathbb{Z} \cong \coprod_{p \text{ prime}} \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}$
   - $\widehat{\mathbb{Z}} \cong \prod_{p \text{ prime}} \mathbb{Z}_p$
   - $\operatorname{Hom}(\mathbb{Z}[\frac{1}{p}]/\mathbb{Z}, \mathbb{Z}[\frac{1}{p}]/\mathbb{Z}) \cong \mathbb{Z}_p$
   - $\operatorname{Hom}(\mathbb{Q}/\mathbb{Z}, \mathbb{Q}/\mathbb{Z}) \cong \widehat{\mathbb{Z}}$

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2 Categorical constructions

- A **discrete category** is a set $\operatorname{ob}\mathcal{C} = X$ with $\operatorname{Hom}_\mathcal{C}(x,y) = \varnothing$ for $x \neq y$ and $\operatorname{Hom}_\mathcal{C}(x,x) = \{\operatorname{id}_x\}$.
- A **groupoid** is a category $\mathcal{C}$ such that every morphism in $\mathcal{C}$ is an isomorphism.
- A **connected groupoid** is a groupoid where any two objects are isomorphic.
- Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor, and $y \in \mathcal{D}$. We define the category $(y \Rightarrow F)$ whose objects are pairs (x, f), where $x \in \mathcal{C}$ and $f\colon y \to F(x)$, and whose morphisms $(x_1, f_1) \to (x_2, f_2)$ are given by morphisms $x_1 \to x_2$ in $\mathcal{C}$ making the obvious diagram commute.
- If $\mathcal{D}$ is a concrete category (i.e. its objects have sub-objects) we define $(* \Rightarrow F)$ to be the category who objects are pairs (x, y), with $x \in \mathcal{C}$ and $y \in F(x)$.
- Given a category $\mathcal{C}$ we define the **presheaf category** $\operatorname{PShv}(\mathcal{C})$ as $\operatorname{Fun}(\mathcal{C}^{\text{op}}, \mathsf{Set})$.

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3 Types of categories

A category $\mathcal{C}$ is ...

- **small** if both $\operatorname{ob}\mathcal{C}$ and $\operatorname{Hom}_{\mathcal{C}}(x, y)$ are proper sets (for all $x, y \in \mathcal{C}$).
- **locally small** if $\operatorname{Hom}_{\mathcal{C}}(x, y)$ is a proper set for all $x, y \in \mathcal{C}$.
- **essentially small** if it equivalent to a small category.
- **locally presentable** if there exists an essentially small subcategory $\mathcal{C}^c \hookrightarrow \mathcal{C}$ of so-called **compact objects** such that the functor $\mathcal{C} \to \operatorname{Fun}_{\mathsf{lex}}((\mathcal{C}^c), \mathsf{Set}^{\mathsf{op}}$ is an equivalence – or equivalently if $\mathcal{C}$ has a fully faithful embedding $i: \mathcal{C} \hookrightarrow \operatorname{PShv}(\mathcal{D})$ to some essentially small category $\mathcal{D}$ such that i commutes with filtered colimits and has a left adjoint.
- **total** if the Yoneda embedding $\mathcal{C} \to \mathsf{PShv}(\mathcal{C})$ has a left adjoint.
- **cofiltered** if it has cones for every finite diagram or equivalently if satisfies the following three properties:
  1. it is non-empty;
  2. for every $i, j \in \mathcal{C}$ there exists $k \in \mathcal{C}$ with $k \to i$ and $k \to j$;
  3. for every $i, j \in \mathcal{C}$ and $f, g: i \to j$ there exists $k \in \mathcal{C}$ with $k \to i$ such that $k \to i \rightrightarrows j$ commutes.
- **filtered** if $\mathcal{C}^{\mathsf{op}}$ is cofiltered.
- *Let $\mathcal{C}$ be a locally small category with small colimits. Then $\mathcal{C}$ has a final object iff the **solution set condition** is satisfied: there exists a proper set I and a family $\{c_i\}_{i \in I}$ of objects of $\mathcal{C}$ such that for every $x \in \mathcal{C}$ there exists an $i \in I$ such that $\operatorname{Hom}_{\mathcal{C}}(x, c_i)$ is non-empty.*

  If $\mathcal{C}$ has a final object then take c to be the final object (and $I = \{*\}$). If $\mathcal{C}$ satisfies the solution set condition then let $w = \coprod_{i \in I} c_i$. We claim that the coequaliser c of *all endomorphisms* of w is final, i.e. that $\operatorname{Hom}(x, c)$ has a single element (up to isomorphism). Note that $c \rightarrowtail w$ is a regular monomorphism.

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4 Interaction between functors and limits

4.1 (Left/Right-)Exact functors

4.1.1 Definitions

- A functor $F: \mathcal{C} \to \mathcal{D}$ is **left exact** if it preserves *finite limits*; it is **right exact** if it preserves *finite colimits*.
- If $F: \mathcal{C} \to \mathcal{D}$ is an *additive* functor between *abelian* categories and $0 \to A \to B \to C \to 0$ is a *short exact sequence* in $\mathcal{C}$ then we have the following equivalent definition: F is **left exact** if $0 \to F(A) \to F(B) \to F(C)$ is exact; it is **right exact** if $F(A) \to F(B) \to F(C) \to 0$ is exact.
- A functor $F: \mathcal{C} \to \mathcal{D}$ is **exact** if it is left exact and right exact.

4.1.2 Properties

Let $F: \mathcal{C} \to \mathcal{D}$ be an exact functor. Then F preserves long exact sequences.

Prove this.

4.2 Interaction

Let $F: \mathcal{C} \to \mathcal{D}$ be a functor. It is simple to show that F sends cones on $M: I \to \mathcal{C}$ to cones on $F \circ M$, but it might not preserve **universal cones**, i.e. limits.

- We say that F **preserves limits** if for any diagram $M: I \to \mathcal{C}$ the functor F sends a universal cone to a universal cone, or equivalently if the canonical morphism $F(\lim_I M) \to \lim_I(F \circ J)$ is an isomorphism.

- We say that F **reflects limits** if for any diagram $M\colon I \to \mathcal{C}$, any cone which is sent to a universal cone is itself universal.
- We say that F **creates limits** if for any diagram $M\colon I \to \mathcal{C}$, the preimage of any universal cone on $F \circ M$ is non-empty and consists of universal cones.
- *If limits of shape I exist in $\mathcal{D}$ and F creates limits of shape I then limits of shape I exist in $\mathcal{C}$ and F preserves and reflects them.*

1. *Let $\mathcal{C}$ be a locally small category. Then* $\mathrm{Hom}_{\mathcal{C}}(x,-)\colon \mathcal{C} \to \mathsf{Set}$ *preserves limits for all $x \in \mathcal{C}$, and dually for $\mathrm{Hom}_{\mathcal{C}}(-,x)$, i.e.*
$$\mathrm{Hom}_{\mathcal{C}}(x, \lim_I M) \cong \lim_I \mathrm{Hom}(x, M)$$
$$\mathrm{Hom}_{\mathcal{C}}(\operatorname{colim}_I M, x) \cong \lim_I \mathrm{Hom}(M, x)$$
We have the following chain of natural (in x) isomorphisms:
$$\begin{aligned}\mathrm{Hom}_{\mathsf{Set}}(\{*\}, \mathrm{Hom}_{\mathcal{C}}(x, \lim_I M)) &\cong \mathrm{Hom}_{\mathcal{C}}(x, \lim_I M)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathcal{C})}(\Delta_x, M)\\ &\cong \mathrm{Hom}_{\mathrm{Fun}(I,\mathsf{Set})}(\Delta_{\{*\}}, \mathrm{Hom}_{\mathcal{C}}(x, M))\\ &\cong \mathrm{Hom}_{\mathsf{Set}}(\{*\}, \lim_I \mathrm{Hom}_{\mathcal{C}}(x, M)).\end{aligned}$$
2. *Let $F\colon \mathcal{C} \rightleftarrows \mathcal{D}\colon G$ be an adjunction $(F \dashv G)$. Then G preserves* limits *and F preserves* colimits.
This follows from the Hom definition of adjunctions and the previous statement about Hom preserving limits.
3. *If $(F \dashv G)$ then in particular F is right-exact and G is left-exact.*
4. *Let $\mathcal{C}$ be locally presentable and $\mathcal{D}$ locally small. Then $F\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint iff it preserves small colimits.*
Work through the proof of this (p. 37 of lecture notes).
5. *Let $\mathcal{C}$ be locally presentable and total and $\mathcal{D}$ locally presentable. Then $F\colon \mathcal{C} \to \mathcal{D}$ has a right adjoint iff it preserves colimits.*
Work through the proof of this (p. 36 of lecture notes).

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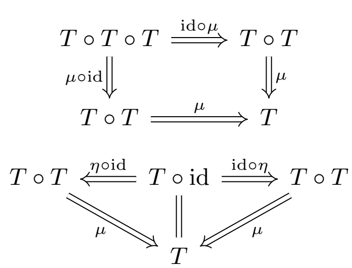
5 Monads

5.1 (Co)Monads

Let $\mathcal{C}$ be a category.

- A **monad** T **acting on** $\mathcal{C}$ is an endofunctor $T\colon \mathcal{C} \to \mathcal{C}$ together with two natural transformations: **multiplication** $\mu\colon T \circ T \Rightarrow T$ and **unit** $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow T$ such that the following diagrams commute:



- A **comonad** T **acting on** $\mathcal{C}$ is a monad acting on $\mathcal{C}^{\mathrm{op}}$, and we call the natural transformations **comultiplication** $\Delta\colon T \Rightarrow T \circ T$ and **counit** $\epsilon\colon T \Rightarrow \mathrm{id}_{\mathcal{C}}$.
- *Let $F\colon \mathcal{C} \leftrightarrows \mathcal{D}\colon G$ be an adjunction $F \dashv G$. Then GF has the structure of a monad on $\mathcal{C}$ and FG has the structure of a comonad on $\mathcal{D}$.*
This is a diagram check. For $T = GF$ the unit of T is the unit of the adjunction $e\colon \mathrm{id}_{\mathcal{C}} \Rightarrow GF$ and the multiplication of T is induced by the counit $\epsilon\colon FG \Rightarrow \mathrm{id}_{\mathcal{D}}$, i.e. $\mu\colon GFGF \Rightarrow GF$ is given by $\mu = \mathrm{id}_G \circ \epsilon \circ \mathrm{id}_F$.

5.2 Algebras

Let $T\colon \mathcal{C} \to \mathcal{C}$ be a monad on $\mathcal{C}$.

- An **algebra over** T is an object $x \in \mathcal{C}$ together with a morphism $a\colon Tx \to x$ such that
  1. the composite $x \xrightarrow{\eta_x} Tx \xrightarrow{a} x$ is the identity;
  2. the following diagram commutes:

  $$\begin{array}{ccc} T^2x & \xrightarrow{T(a)} & Tx \\ \downarrow{\scriptstyle \mu_x} & & \downarrow{\scriptstyle a} \\ Tx & \xrightarrow{a} & x \end{array}$$

  and we define **coalgebras** dually.

- We write $\mathsf{Alg}_T(\mathcal{C})$ to denote the **category of T-algebras in** $\mathcal{C}$, whose objects are T-algebras (x, a) and whose morphisms are induced by morphisms $f\colon x \to y$ making the obvious diagram commute.

5.3 (Co)Monadic functors

- *Let T be a monad on $\mathcal{C}$. Then the forgetful functor $R\colon \mathsf{Alg}_T(\mathcal{C}) \to \mathcal{C}$ has a* left *adjoint, and the associated monad on $\mathcal{C}$ is naturally isomorphic to T. Dually, the forgetful functor of S-coalgebras has a* right *adjoint whose associated comonad is isomorphic to S.*

  Given a monad T on $\mathcal{C}$ we define $L\colon \mathcal{C} \to \mathsf{Alg}_T(\mathcal{C})$ by $x \mapsto Tx$, where the T-algebra structure on Tx is given by $T^2x \xrightarrow{\mu_x} Tx$. Now let (A, a_A) be a T-algebra and $x \in \mathcal{C}$. Then a morphism of T-algebras $Tx \to A$ is a morphism $f\colon Tx \to A$ in $\mathcal{C}$ such that the square in the following diagram commutes:

  $$\begin{array}{ccc} Tx & & \\ \downarrow{\scriptstyle T(\eta_x)} & \searrow{=} & \\ T^2x & \xrightarrow[\mu_x]{} & Tx \\ \downarrow{\scriptstyle T(f)} & & \downarrow{\scriptstyle f} \\ TA & \xrightarrow{a_A} & A \end{array}$$

  noting that the triangle commutes by the unit axiom of the monad T. But then f is uniquely determined by the composite $a_A \otimes T(f) \otimes T(\eta_x)$, i.e. by $f \circ \eta_x$. Thus $\mathrm{Hom}_{\mathsf{Alg}_T(\mathcal{C})}(Lx, A) \cong \mathrm{Hom}_{\mathcal{C}}(x, RA)$.

- A functor $G\colon \mathcal{D} \to \mathcal{C}$ is **monadic** if it has a *left* adjoint, and for the corresponding monad T the functor $\mathcal{D} \to \mathsf{Alg}_T(\mathcal{C})$ is an equivalence.

- A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **comonadic** if it has a *right* adjoint, and for the corresponding comonad S the functor $\mathcal{C} \to \mathsf{Coalg}_S(\mathcal{C})$ is an equivalence.

- *Let T be a monad on $\mathcal{C}$. Then the forgetful functor $R\colon \mathsf{Alg}_T(\mathcal{C}) \to \mathcal{C}$ creates* limits*. Further, if T preserves* colimits *of shape I then the forgetful functor reflects those colimits.*

  Let $J\colon I \to \mathsf{Alg}_T(\mathcal{C})$ be a diagram. We need to show that $\lim_I FJ$ has a unique T-algebra structure compatible with the forgetful maps, i.e. we need to construct and show uniqueness of a map $T(\lim_I FJ) \to \lim_I FJ$ that makes the following diagram commute for all $i \in I$:

  $$\begin{array}{ccc} T(\lim_I FJ) & \longrightarrow & \lim_I FJ \\ \downarrow & & \downarrow \\ TFJ(i) & \longrightarrow & FJ(i) \end{array}$$

where the bottom morphism comes from the T-algebra maps on the $J(i)$. By a universal property, every such map uniquely factors through $\lim_I TFJ$ and is thus uniquely determined.

Now suppose that T preserves colimits of shape I and let $J\colon I \to \mathsf{Alg}_T(\mathcal{C})$ be a diagram. We endow $\operatorname{colim}_I FJ$ with a T-algebra structure by defining the action $T(\operatorname{colim}_I FJ) \cong \operatorname{colim}_I TFJ \to \operatorname{colim}_I FJ$. The universal property of this T-algebra follows from the universal property of the colimit.

5.4 Barr-Beck

5.4.1 Conservative functors and split pairs

- A functor $F\colon \mathcal{C} \to \mathcal{D}$ is **conservative** if it reflects isomorphisms, i.e. if f is a morphism in $\mathcal{C}$ such that $F(f)$ is an isomorphism in $\mathcal{D}$ then f is an isomorphism.
- *Let $F\colon \mathcal{C} \to \mathcal{D}$ be a conservative functor which admits a fully faithful adjoint (either left or right). Then F is an equivalence.*

  Assume that F admits a fully faithful left adjoint F^L. Full faithfulness implies that the unit $\eta\colon \mathrm{id}_{\mathcal{D}} \Rightarrow FF^L$ is a natural isomorphism; we need to show that the counit $\epsilon\colon F^L F \Rightarrow \mathrm{id}_{\mathcal{C}}$ is also an isomorphism. One of the adjunction axioms tells us that the composite $F(\epsilon_x) \circ \eta_{F(x)}$ is $\mathrm{id}_{F(x)}$ for all $x \in \mathcal{C}$. Since $\eta_{F(x)}$ is an isomorphism, $F(\epsilon_x)$ must also be an isomorphism. But F is conservative, and so ϵ_x is an isomorphism for all $x \in \mathcal{C}$.
- If we have a pair of morphisms $f, g\colon x \to y$ then a **fork** is a cocone

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \xrightarrow{e} z.$$

- A fork is **split** if there exists morphisms $z \xrightarrow{s} y \xrightarrow{t} x$ such that
  1. $es = \mathrm{id}_z$
  2. $ft = \mathrm{id}_y$
  3. $gt = se$.
- *Every split fork is a coequaliser.*

  Say we have some split fork

$$x \underset{g}{\overset{f}{\rightrightarrows}} y \xrightarrow{e} z.$$

  Let $w \in \mathcal{C}$ come with a morphism $e'\colon y \to w$ such that $e'f = e'g$. Define $h = e's\colon z \to w$, so that $he = e'se = e'gt = e'ft = e'$. Further, h is uniquely defined since $h = hes = e's$.
- A pair of morphisms $f, g\colon x \to y$ is called a **split pair** if their coequaliser exists and is split.
- A pair of morphisms $f, g\colon x \to y$ is called an F**-split pair** for a functor $F\colon \mathcal{C} \to \mathcal{D}$ if $F(f), F(g)$ is a split pair.

5.4.2 Important examples

1. *Let T be a monad on $\mathcal{C}$ with multiplication μ and let $x \in \mathcal{C}$ be a T-algebra with action a. Then we have the split pair*

$$T^2x \underset{T(a)}{\overset{\mu_x}{\rightrightarrows}} Tx.$$

   The splitting is given by $s = \eta_x\colon x \to Tx$ and $t = \eta_{Tx}\colon Tx \to T^2x$.
2. *Let $F \dashv G$ be an adjunction. Then we have the G-split fork*

$$FGFG(y) \underset{FG(\epsilon_y)}{\overset{\epsilon_{FG(y)}}{\rightrightarrows}} FG(y) \xrightarrow{\epsilon_y} y.$$

   Applying G we get a fork which is a split fork for the GF-algebra $G(y)$ by the previous example.

5.4.3 Barr-Beck theorem

- **The monadic Barr-Beck theorem:** *A functor $G\colon \mathcal{D} \to \mathcal{C}$ is monadic iff the following conditions hold:*
  1. *G admits a left adjoint;*
  2. *G is conservative;*
  3. *every G-split pair of morphisms admits a coequaliser in $\mathcal{D}$ and it is preserved by G.*

  Work through a sketch *proof of this (p. 46 of lecture notes).*

- **The comonadic Barr-Beck theorem:** *A functor $F\colon \mathcal{C} \to \mathcal{D}$ is comonadic iff the following conditions hold:*
  1. *F admits a right adjoint;*
  2. *F is conservative;*
  3. *every F-cosplit pair of morphisms admits an equaliser in $\mathcal{C}$ and it is preserved by F.*

5.5 Descent

Work through this section in the lecture notes.

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6 Categorical algebraic geometry

Let k be an (algebraically closed) field (of characteristic zero).

6.1 Affine schemes

- We write Alg_k to mean the **category of k-algebras**.
- A **space** is a functor $\mathsf{Alg}_k \to \mathsf{Set}$. We write $\mathsf{Sp} = \mathrm{Fun}(\mathsf{Alg}_k, \mathsf{Set})$ to mean the **category of spaces**.
- The value of a space on a ring R is called the **set of R-points of the space**.
- The **spectrum functor** Spec is the Yoneda functor $\mathsf{Alg}_k^{\mathrm{op}} \hookrightarrow \mathsf{Sp}$.
- The **category of affine schemes** Aff is defined to be the essential image of the spectrum functor $\mathsf{Alg}_k^{\mathrm{op}} \hookrightarrow \mathsf{Sp}$.
- *By definition then, we have the isomorphism of categories* $\mathsf{Aff} \cong \mathsf{Alg}_k^{op}$. *This lets us rewrite **spaces** as presheaves on affine schemes, i.e.* $\mathsf{Sp} \cong \mathrm{PShv}(\mathsf{Aff})$.
- The **ring of functions** $\mathcal{O}\colon \mathsf{Sp} \to \mathsf{CRing}^{\mathrm{op}}$ is the functor given by $\mathcal{O}(X) = \lim_{\mathrm{Spec}\, R \to X} R$.
- *The functor $\mathcal{O}$ sends colimits in* Sp *to limits in* CRing. *Thus*

$$\mathrm{Spec}\, R_1 \times_{\mathrm{Spec}\, S} \mathrm{Spec}\, R_2 \cong \mathrm{Spec}(R_1 \otimes_S R_2).$$

- Two very important examples of affine schemes are
  1. $\mathbb{A}^n \cong \mathrm{Spec}\, k[x_1, \ldots, x_n]$;
  2. $\mathbb{G}_m \cong \mathrm{Spec}\, k[x, x^{-1}]$, where $\mathbb{G}_m(R) = R^\times$.

6.2 Non-affine schemes

- A module M over a commutative ring R is **invertible** if there exists an R-module N such that $M \otimes_R N \cong R$.
- Define **projective n-space** $\mathbb{P}^n$ to be the functor which maps R to the set of invertible R-submodules of $R^{\oplus(n+1)}$. If R is a field then all invertible modules are isomorphic to R, and $\mathbb{P}^n(R)$ is the set of lines (one-dimensional subspaces) in R^{n+1}, which agrees with the classical notion of a projective space.
- *$\mathbb{P}^n$ is not an affine scheme. In particular, $\mathcal{O}(\mathbb{P}^n) \cong k$.*

6.3 Tangent spaces

Let $X \in \mathsf{Sp}$ be a space and $p \in X(k)$ be a k-point.

- Write $\pi \colon k[\varepsilon]/\varepsilon^2 \to k$ **to mean** the natural projection.
- The **tangent space** T_pX **of** X **at** p is the fibre of $X(\pi)$ at $p \in X(k)$.

6.4 Quasi-coherent sheaves

Let $X \in \mathsf{Sp}$ be a space.

- A **quasi-coherent sheaf** $\mathcal{F}$ **on** X is the following collection of data:
  1. an R-module $f^*\mathcal{F}$ for every ring R and morphism $f \colon \operatorname{Spec} R \to X$;
  2. an isomorphism $f^*\mathcal{F} \otimes_{R_2} R_1 \cong (f \circ g)^*\mathcal{F}$ for every morphism $g \colon \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ of affine schemes and $f \colon \operatorname{Spec} R_2 \to X$,

  where the isomorphisms satisfy the **cocycle condition**: for any pair of morphisms $g_1 \colon \operatorname{Spec} R_1 \to \operatorname{Spec} R_2$ and $g_2 \colon \operatorname{Spec} R_2 \to \operatorname{Spec} R_3$ and morphism $f \colon \operatorname{Spec} R_3 \to X$ we have equality between the two following isomorphisms:
  - $(f \circ g_2 \circ g_1)^*\mathcal{F} \cong f^*\mathcal{F} \otimes_{R_3} R_1$
  - $(f \circ g_2 \circ g_1)^*\mathcal{F} \cong (f \circ g_2)^*\mathcal{F} \otimes_{R_2} R_1 \cong f^*\mathcal{F} \otimes_{R_3} R_2 \otimes_{R_2} R_1 \cong f^*\mathcal{F} \otimes_{R_3} R_1$

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7 Tensor product

7.1 Definition

7.1.1 Modules over a ring

The tensor product $A \otimes_R B$ of two modules A and B over a *commutative* ring R is defined as

$$A \otimes_R B \cong F(A \times B)/G$$

where $F(A \times B)$ is the free R-module generated by the cartesian product, and G is the R-module generated by the relations

1. $(a,b) + (a',b) \sim (a+a',b)$;
2. $(a,b) + (a,b') \sim (a,b+b')$;
3. $r(a,b) \sim (ar,b) \sim (a,rb)$.

7.1.2 Categorical

The bilinear map $\vartheta \colon A \times B \to A \otimes_R B$ given by $\vartheta \colon (a,b) \mapsto a \otimes_R b$ is such that any other *bilinear* map $A \times B \to W$ for any W uniquely factors through ϑ.

7.2 Properties

We write $\otimes$ to mean $\otimes_R$, and we write $\coprod$ instead of the more common $\oplus$ to avoid notational similarities. We write M, N, P to mean arbitrary R-modules, and I, J to mean ideals of R.

- $R \otimes M \cong M$
- $M \otimes (N \otimes P) \cong (M \otimes N) \otimes P$
- $M \otimes N \cong N \otimes M$

- $M \otimes \coprod_{i \cong 1}^n N_i \cong \coprod_{i \cong 1}^n M \otimes N_i$
- (so in particular $M \otimes R^n \cong M^n$)

- $(- \otimes M) \dashv \operatorname{Hom}(M, -)$

- (so tensoring is *right exact* and thus *commutes with colimits*)
- (so in particular $(\coprod_{i\in I} N_i) \otimes M \cong \coprod_{i\in I}(N_i \otimes M)$)

- $R/I \otimes M \cong M/IM$
- if M is flat then $I \otimes M \cong IM$
- $(M/IM) \otimes_{R/I} (N/IN) \cong M \otimes N \otimes R/I$
- $R/I \otimes R/J \cong R/(I+J)$

7.3 Abelian groups ($R = \mathbb{Z}$)

We write A to mean an arbitrary abelian group.

1. $A \otimes \mathbb{Z}/n\mathbb{Z} \cong A/mA$
2. $\mathbb{Z}/m\mathbb{Z} \otimes \mathbb{Z}/n\mathbb{Z} \cong \mathbb{Z}/\gcd(m,n)\mathbb{Z}$
3. $\mathbb{Q} \otimes B \cong \mathbb{Q} \otimes (B/\mathrm{tors}(B))$

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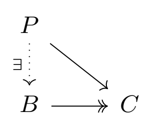
8 Projectives, injectives, flats, and frees

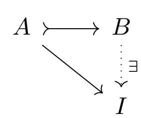
8.1 Definitions

- An object $P \in R$-mod is **projective** if $\mathrm{Hom}(P, -)$ is exact, or equivalently if for every morphism $P \to C$ and epimorphism $B \twoheadrightarrow C$ there exists $P \to B$ such that the following diagram commutes:



  Note that we don't require uniqueness!

- An object $I \in R$-mod is **injective** if $\mathrm{Hom}(-, I)$ is exact, or equivalently if for every morphism $A \to I$ and monomorphism $A \rightarrowtail B$ there exists $B \to I$ such that the following diagram commutes:



- We say that R-mod **has enough projectives** if for every object $A \in R$-mod there exists a projective $P \in R$-mod with an epimorphism $P \twoheadrightarrow A$.
- We say that R-mod **has enough injectives** if for every object $A \in R$-mod there exists an injective $I \in R$-mod with a monomorphism $A \rightarrowtail I$.
- An object $F \in R$-mod is **flat** if $- \otimes_R F$ is exact, or equivalently if for every $A \rightarrowtail B$ the corresponding map $A \otimes_R F \to B \otimes_R F$ is a monomorphism.

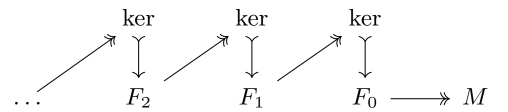
8.2 Properties and equivalent definitions

1. *An R-module projective iff it's the direct summand of a free module.*
2. *Any free module is projective.*
3. *In* Ab *we have the following classifications:*
   - *projective modules are exactly the free modules;*
   - *injective modules are exactly the divisible groups;*
   - *flat modules are exactly the torsion-free groups.*

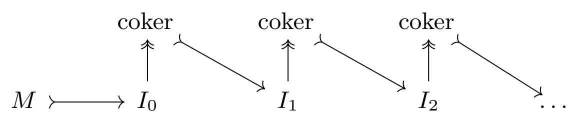
HA 9 Resolutions

Let $M \in R$-mod.

- A **projective** (respectively, **free**) **resolution of** M is a *chain* complex $(P_\bullet, d_\bullet)$ such that
  - each P_n is projective (respectively, free);
  - $P_n = 0$ for $n \leqslant -1$;
  - $H_n(P_\bullet) = \delta_{n0} M$.
- If we have enough projectives then we can construct the canonical free resolution as follows:



- An **injective resolution of** M is a *cochain* complex $(I^\bullet, d^\bullet)$ such that
  - each I_n is projective;
  - $I_n = 0$ for $n \leqslant -1$;
  - $H^n(I^\bullet) = \delta_{n0} M$.
- If we have enough injectives then we can construct the canonical injective resolution as follows:



- Another way of thinking of a resolution is as a **quasi-isomorphism** i.e. a chain map that induces an isomorphism in homology. Then a **projective** (respectively, **flat**) **resolution** is a quasi-isomorphism $P_\bullet \to M[0]$ (respectively, $F_\bullet \to M[0]$), where $M[0]$ is the chain complex concentrated as M in degree 0. An **injective resolution** is a quasi-isomorphism $M[0] \to I^\bullet$. Diagrammatically, e.g. a projective resolution is an *exact* sequence

$$\begin{array}{ccccccc} \ldots \longrightarrow & P_2 & \longrightarrow & P_1 & \longrightarrow & P_0 & \\ & & & & & \downarrow & \\ & & & & & M & \longrightarrow 0 \end{array}$$

HA AT 10 Computing Tor and Ext

10.1 Definitions

We assume that R is a commutative ring (with unity).

Let $M, N \in R$-mod with projective resolutions $P_\bullet \twoheadrightarrow M$ and $Q_\bullet \twoheadrightarrow N$, and injective resolution $N \hookrightarrow J^\bullet$. We make the following definitions.

$$\mathrm{Tor}_n^R(M,N) = \begin{cases} H_n(P_\bullet \otimes N) \\ H_n(M \otimes Q_\bullet) \end{cases} \qquad \mathrm{Tor}_\bullet^R(M,N) = \bigoplus_{n \geqslant 0} \mathrm{Tor}_n^R(M,N)$$

$$\mathrm{Ext}_R^n(M,N) = \begin{cases} H^n(\mathrm{Hom}(P_\bullet, N)) \\ H^n(\mathrm{Hom}(M, J^\bullet)) \end{cases} \qquad \mathrm{Ext}_R^\bullet(M,N) = \bigoplus_{n \geqslant 0} \mathrm{Ext}_R^n(M,N)$$

Don't get confused by degrees: We follow the convention of writing *chain* complexes (the differential *decreases* degree) as $\ldots \to \ldots$ and *cochain* (the differential *increases* degree) as $\ldots \leftarrow \ldots$, so that the degrees line up if we write one below the other. Remember that $\mathrm{Hom}(-, B)$ is *contravariant* and $\mathrm{Hom}(A, -)$ is *covariant*, and so both $\mathrm{Hom}(C_\bullet, B)$ and $\mathrm{Hom}(A, D^\bullet)$ will be *cochain* complexes:

$$\begin{array}{ccccccccc} \ldots & \to & C_2 & \to & C_1 & \to & C_0 & \to & 0 \\ \ldots & \leftarrow & \mathrm{Hom}(C_2, B) & \leftarrow & \mathrm{Hom}(C_1, B) & \leftarrow & \mathrm{Hom}(C_0, B) & \leftarrow & 0 \end{array}$$

$$\begin{array}{ccccccccc} \ldots & \leftarrow & D_2 & \leftarrow & D_1 & \leftarrow & D_0 & \leftarrow & 0 \\ \ldots & \leftarrow & \mathrm{Hom}(A, D_2) & \leftarrow & \mathrm{Hom}(A, D_1) & \leftarrow & \mathrm{Hom}(A, D_0) & \leftarrow & 0 \end{array}$$

10.2 Useful facts

Note that $\mathrm{Tor}_0(M, N) \cong M \otimes N$ *and* $\mathrm{Ext}^0(M, N) \cong \mathrm{Hom}(M, N)$*, so we are usually only interested in* Tor_n *and* Ext^n *for* $n \geqslant 1$.

- M is projective $\implies \mathrm{Ext}^n(M, -) = 0$ for $n \geqslant 1$
- M is projective $\iff \mathrm{Ext}^1(M, -) = 0$

- N is injective $\implies \mathrm{Ext}^n(-, N) = 0$ for $n \geqslant 1$
- N is injective $\iff \mathrm{Ext}^1(-, N) = 0$

- M is flat $\implies \mathrm{Tor}_\bullet(\mathrm{M}, \mathrm{N}) = 0$
- M is flat $\iff \mathrm{Tor}_1(M, -) = 0$

- $\mathrm{Tor}_n(\mathrm{colim}_\alpha M_\alpha, N) \cong \mathrm{colim}_\alpha \mathrm{Tor}_n(M_\alpha, N)$
- $\mathrm{Tor}_n(M, N) \cong \mathrm{Tor}_n(N, M)$
- $\mathrm{Ext}^n(\mathrm{colim}_\alpha M_\alpha, N) \cong \lim_\alpha \mathrm{Ext}^n(M_\alpha, N)$
- $\mathrm{Ext}^n(M, \lim_\beta N_\beta) \cong \lim_\beta \mathrm{Ext}(M, N_\beta)$

10.3 Dimension shifting

10.3.1 Tor

Let $X, Y \in R\text{-mod}$. Suppose that

$$0 \to Y \to P_k \to P_{k-1} \to \ldots \to P_0 \to X \to 0$$

is an *exact sequence* in R-mod, with each P_i *projective. Then for any* $N \in R$-mod *there is a canonical isomorphism*

$$\mathrm{Tor}_n^R(X, N) \cong \mathrm{Tor}_{n-k-1}^R(Y, N).$$

10.3.2 Ext

Let $X, Y \in R$-mod.

1. Suppose that
   $$0 \leftarrow Y \leftarrow I^k \leftarrow I^{k-1} \leftarrow \ldots \leftarrow I^0 \leftarrow X \leftarrow 0$$
   is an *exact* sequence in R-mod, with each I^i *injective. Then for any* $M \in R$-mod *there is a canonical isomorphism*
   $$\mathrm{Ext}_R^n(M, X) \cong \mathrm{Ext}_R^{n+k+1}(M, Y).$$
2. Suppose that
   $$0 \to Y \to P_k \to P_{k-1} \to \ldots \to P_0 \to X \to 0$$
   is an *exact sequence* in R-mod, with each P_i *projective. Then for any* $N \in R$-mod *there is a canonical isomorphism*
   $$\mathrm{Ext}_R^n(X, N) \cong \mathrm{Ext}_R^{n+k+1}(Y, N).$$

Check the above statements.

10.4 Long exact sequences

Let $0 \to A \to B \to C \to 0$ be a short exact sequence in R-mod, and N some arbitrary object in R-mod. Then Tor and Ext give rise to the following long exact sequences:

1. **Tor(–, N):** *Recall that* $\mathrm{Tor}_0^R(M, N) \cong M \otimes_R N$.

$$
\begin{array}{ccccc}
 & & & & \cdots \\
\to \mathrm{Tor}_n^R(A,N) & \longrightarrow & \mathrm{Tor}_n^R(B,N) & \longrightarrow & \mathrm{Tor}_n^R(C,N) \\
\to \mathrm{Tor}_{n-1}^R(A,N) & \longrightarrow & \mathrm{Tor}_{n-1}^R(B,N) & \longrightarrow & \mathrm{Tor}_{n-1}^R(C,N) \\
\cdots & & \cdots & & \cdots \\
\to A \otimes_R N & \longrightarrow & B \otimes_R N & \longrightarrow & C \otimes_R N \longrightarrow 0
\end{array}
$$

2. **Ext(–, N):** *Recall that* $\mathrm{Ext}_R^0(M, N) \cong \mathrm{Hom}_{R\text{-mod}}(M, N)$.

$$
\begin{array}{ccccc}
 & & & & \cdots \\
\mathrm{Ext}_R^{n+1}(A,N) & \longleftarrow & \mathrm{Ext}_R^{n+1}(B,N) & \longleftarrow & \mathrm{Ext}_R^{n+1}(C,N) \leftarrow \\
\mathrm{Ext}_R^{n}(A,N) & \longleftarrow & \mathrm{Ext}_R^{n}(B,N) & \longleftarrow & \mathrm{Ext}_R^{n}(C,N) \leftarrow \\
\cdots & & \cdots & & \cdots \\
\mathrm{Hom}(A,N) & \longleftarrow & \mathrm{Hom}(B,N) & \longleftarrow & \mathrm{Hom}(C,N) \longleftarrow 0
\end{array}
$$

3. *The same sequence for* **Ext(M, –)** *also holds.*

10.5 Abelian groups ($R = \mathbb{Z}$)

Recall that, when $R = \mathbb{Z}$, projective = free, injective = divisible, and flat = torsion-free. We write $\mathrm{tors}(A)$ *to mean the torsion subgroup of A.*

Claim: $\mathbb{Z}$ *and* $\mathbb{Q}$ *are flat* $\mathbb{Z}$*-modules.*
Proof: They are both torsion-free.

10.5.1 Examples of $\mathrm{Tor}_n^{\mathbb{Z}}(A, B)$

Claim: $\mathrm{Tor}_n^{\mathbb{Z}}(A, B) = 0$ *for any abelian groups A, B when $n \geqslant 2$.*
Proof: Take the generators of A and consider the free abelian group F that they generate. Then we have the epimorphism $F \twoheadrightarrow A$ with kernel $K \subset F$. Since any subgroup of a free abelian group is free ($\mathbb{Z}$ is a PID) we have the short exact sequence

$$0 \to K \hookrightarrow F \twoheadrightarrow A \to 0$$

with K, F both free, and thus projective, and so this is a projective resolution of A that is zero in degree n for $n \geqslant 2$. This means that it will also have zero homology in degree n for $n \geqslant 2$.

Let A be an arbitrary abelian group.

1. $\mathrm{Tor}_1^{\mathbb{Z}}(A, \mathbb{Z}) = 0$
   Use that $\mathbb{Z}$ *is flat.*

2. $\mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q}) = 0$
   Use that $\mathbb{Q}$ is flat.
3. $\mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Z}/m\mathbb{Z}) \cong \{a \in A \mid ma = 0\}$
   Use the projective resolution $(0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z})$ for $\mathbb{Z}/m\mathbb{Z}$. Since $(- \otimes \mathbb{Z}) \cong \mathrm{id}$ we obtain the sequence $(0 \to A \xrightarrow{\cdot m} A \to 0)$, which has homology

$$\ker(A \xrightarrow{\cdot m} A) = \{a \in A \mid ma = 0\}$$

   in degree 1, as claimed.
4. $\mathrm{Tor}_1^{\mathbb{Z}}(A,\mathbb{Q}/\mathbb{Z}) \cong \mathrm{tors}(A)$
   Use the projective resolution $(0 \to \mathbb{Z} \xrightarrow{\iota} \mathbb{Q})$ for $\mathbb{Q}/\mathbb{Z}$. Then applying $(A \otimes -)$ gives

$$0 \to A \xrightarrow{\iota} A \otimes \mathbb{Q} \to 0.$$

   If A is torsion then $A \otimes \mathbb{Q} = 0$, and so the homology in degree 1 is $A = \mathrm{tors}(A)$. If A is torsion-free then $A \hookrightarrow A \otimes \mathbb{Q}$, and so the homology in degree 1 is $0 = \mathrm{tors}(A)$. Generally, we see that $\ker(A \xrightarrow{\iota} A \otimes \mathbb{Q}) \cong \mathrm{tors}(A)$.

Compute some examples using $\mathbb{Z}[p^{-1}]$, $\mathbb{Z}/p^\infty$, and $\hat{\mathbb{Z}}$.

10.5.2 Examples of $\mathrm{Ext}_{\mathbb{Z}}^n(A,B)$

Claim: $\mathrm{Ext}_{\mathbb{Z}}^n(A,B) = 0$ *for any abelian groups A, B when $n \geqslant 2$.*

Proof: Similar to the equivalent statement for Tor, noting that the quotient of a divisible group is divisible.

Let A, B be arbitrary abelian groups.

1. $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z},B) = 0$
   Use the fact that $\mathbb{Z}$ is projective.
2. $\mathrm{Ext}_{\mathbb{Z}}^1(A,\mathbb{Q}) = 0$
   Use the fact that $\mathbb{Q}$ is injective.
3. $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Z}/m\mathbb{Z},B) \cong B/mB$
   Use the standard projective resolution $(0 \to \mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z})$ of $\mathbb{Z}/m\mathbb{Z}$. Then, applying $\mathrm{Hom}(-,B)$ (which is contravariant*) we obtain the cochain complex*

$$0 \leftarrow \mathrm{Hom}(\mathbb{Z},B) \xleftarrow{-\circ(\cdot m)} \mathrm{Hom}(\mathbb{Z},B) \leftarrow 0.$$

   But $\mathrm{Hom}(\mathbb{Z},B) \cong B$ for any abelian group B, and the induced map $\xleftarrow{\cdot m}$ is still multiplication by m. Thus the homology in degree 1 is

$$\frac{\ker(0 \leftarrow B)}{\mathrm{im}(B \xleftarrow{\cdot m} B)} \cong \frac{B}{mB}$$

   as claimed.

Can you compute $\mathrm{Ext}_{\mathbb{Z}}^1(A,\mathbb{Z}/m\mathbb{Z})$?

Show that $\mathrm{Ext}_{\mathbb{Z}}^1(\mathbb{Q},\mathbb{Z}) \cong (\hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q})/\mathbb{Q}$. *See Section 1.5.5.*

10.6 Modules over polynomial rings in one variable ($R = k[x]$)

*Recall that, in any R-*mod*, projective = direct summand of free, and that R is always a flat R-module.*

Prove that $k[x,x^{-1}]$ is a flat $k[x]$-module by constructing it as a colimit.

10.6.1 Examples of $\mathrm{Tor}_n^{k[x]}(M,N)$

These examples mirror those of abelian groups in Section 10.5.1.
Let M be an arbitrary $k[x]$-module.

1. $\mathrm{Tor}_n^{k[x]}(M,k[x])=0$
   Use that $k[x]$ is flat.
2. $\mathrm{Tor}_n^{k[x]}(M,k[x,x^{-1}])=0$
   Use that $k[x,x^{-1}]$ is flat.

Calculate the following modules:

1. $\mathrm{Tor}_n^{k[x]}\left(\frac{k[x]}{(x^m)},\frac{k[x]}{(x^n)}\right)$
2. $\mathrm{Tor}_n^{k[x]}\left(\frac{k[x,x^{-1}]}{k[x]},\frac{k[x,x^{-1}]}{k[x]}\right)$
3. $\mathrm{Tor}_n^{k[x]}\left(\frac{k[x]}{(x^m)},\frac{k[x,x^{-1}]}{k[x]}\right)$
4. $\mathrm{Tor}_n^{k[x]}(k,k)$

10.6.2 Examples of $\mathrm{Ext}^n_{k[x]}(M,N)$

Compute some examples.

10.7 Modules over $R=k[x]/(x^n)$ and $R=k[x,y]$

Calculate the following modules:

1. $\mathrm{Tor}_n^{k[x]}(k,k)$
2. $\mathrm{Tor}_n^{k[x]/(x^2)}(k,k)$
3. $\mathrm{Tor}_n^{k[x]/(x^3)}(k,k)$
4. $\mathrm{Tor}_n^{k[x]/(x^n)}(k,k)$
5. $\mathrm{Tor}_n^{k[x,y]}(k,k)$
6. $\mathrm{Tor}_n^{k[x,y]}(k[x,y]/(x,y),k[x,y]/(x-a,y-b))$

10.8 Other examples

Let p be a prime and $G=\langle\omega\mid\omega^p=1\rangle$ the cyclic group of order p. Define the $k[G]$-module $S=k[\langle 1+\mu+\ldots+\mu^{p-1}\rangle]$. Find $\mathrm{Ext}^n_{k[G]}(S,S)$.

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11 Working in abelian categories

11.1 Kernels and cokernels

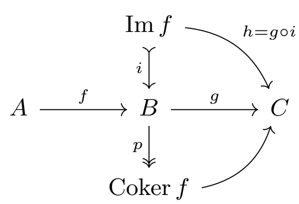
See Table 2.

	Statement	Motto
1.	$\ker \operatorname{Coker}(A \rightarrowtail B) \cong A \rightarrowtail B$ $\operatorname{coker} \operatorname{Ker}(A \twoheadrightarrow B) \cong A \twoheadrightarrow B$	monos are their own image epis are their own coimage
2.	$\operatorname{Ker}(A \to B) \rightarrowtail A$ $B \twoheadrightarrow \operatorname{Coker}(A \to B)$	kernels are mono cokers are epi
3.	$\operatorname{Im}(A \to B) \rightarrowtail B$ $A \twoheadrightarrow \operatorname{Coim}(A \to B)$	images are mono into target sources are epi onto coimages
4.	$\operatorname{Im}(A \to B) \cong B \iff \operatorname{Coker}(A \to B) = 0$ $\operatorname{Coim}(A \to B) \cong A \iff \operatorname{Ker}(A \to B) = 0$	target is image iff trivial cokernel source is coimage iff trivial kernel
5.	$A \twoheadrightarrow B \iff \operatorname{Im}(A \to B) \cong B$ $A \rightarrowtail B \iff \operatorname{Ker}(A \to B) = 0$	epi iff target is image mono iff kernel is trivial
6.	$A \twoheadrightarrow (\operatorname{Im} A \to B)$ $\operatorname{Coim}(A \to B) \rightarrowtail B$ $\operatorname{Im}(A \to B) \cong \operatorname{Coim}(A \to B)$	$A \xrightarrow{f} B$ factors uniquely as $A \twoheadrightarrow \operatorname{Im} f \rightarrowtail B$

Table 2: Let $\mathcal{C}$ be an abelian category, $A, B \in \mathcal{C}$ arbitrary objects, and $f\colon A \to B$ an arbitrary morphism.

11.2 Homology

Let $\mathcal{C}$ be an arbitrary category, and $A \xrightarrow{f} B \xrightarrow{g} C$ in $\mathcal{C}$ such that $g \circ f = 0$. Then there exist i, p such that the following diagram commutes:

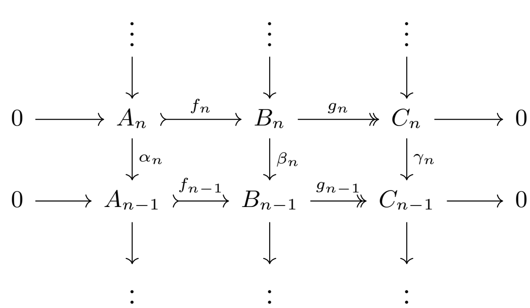


Commutativity implies that $p \circ i = 0$, thus $h = 0$. So there exists a unique $\varphi\colon \operatorname{Im} f \to \operatorname{Ker} g$.

Define the **homology at** B as $\operatorname{Coker} \varphi$.

11.3 Zig-zag lemma

Let $\mathcal{C}$ be an abelian category and $(0 \to A_\bullet \to B_\bullet \to C_\bullet \to 0)$ a chain complex of short exact sequences. That is, the following diagram *commutes*, has *exact rows*, and the *columns are chain complexes*:



Then for all n there exists $d_n\colon H_n(C_\bullet) \to H_{n-1}(A_\bullet)$ such that the following sequence is exact:

$$\dots \to H_n(A_\bullet) \xrightarrow{f_*} H_n(B_\bullet) \xrightarrow{g_*} H_n(C_\bullet) \xrightarrow{d_n} H_{n-1}(A_\bullet) \xrightarrow{f_*} H_{n-1}(B_\bullet) \xrightarrow{g_*} H_{n-1}(C_\bullet) \to \dots$$

11.4 Five lemma

Let $\mathcal{C}$ be an abelian category. Assume that the following diagram *commutes* and has *exact rows*:

$$\begin{array}{ccccccccc} A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & D & \longrightarrow & E \\ \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \downarrow \delta & & \downarrow \varepsilon \\ A' & \longrightarrow & B' & \longrightarrow & C' & \longrightarrow & D' & \longrightarrow & E' \end{array}$$

(i.e. β and δ are isomorphisms, α is an epimorphism, and ε is a monomorphism)

Then γ is an isomorphism.

11.5 Snake lemma

Let $\mathcal{C}$ be an abelian category. Assume that the following diagram *commutes* and has *exact rows*:

$$\begin{array}{ccccccccc} & & A & \longrightarrow & B & \longrightarrow & C & \longrightarrow & 0 \\ & & \downarrow \alpha & & \downarrow \beta & & \downarrow \gamma & & \\ 0 & \longrightarrow & A' & \longrightarrow & B' & \longrightarrow & C' & & \end{array}$$

Then there exists $d\colon \operatorname{Ker}\gamma \to \operatorname{Coker}\alpha$ such that the following sequence is exact:

$$\operatorname{Ker}\alpha \to \operatorname{Ker}\beta \to \operatorname{Ker}\gamma \xrightarrow{d} \operatorname{Coker}\alpha \to \operatorname{Coker}\beta \to \operatorname{Coker}\gamma.$$

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12 Retracts

Retraction: A *retraction of X onto $A \subseteq X$* is a (continuous) map $r\colon X \to A$ such that $r|_A = \mathrm{id}_A$.

Deformation retract: A *deformation retract of X onto $A \subseteq X$* is a (continuous) family of maps $\{r_t\colon X \to X \mid t \in [0,1]\}$ such that $r_0 = \mathrm{id}_X$, $r_1(X) = A$, and $r_t|_A = \mathrm{id}_A$ for all t. That is, it is a homotopy from the identity on X to a retract of X onto A.

For example, every space X has a *retract* onto any point $x \in X$, but there is a *deformation retract* only if (not if and only if) X is path-connected.

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13 Homology

13.1 Simplicial and singular

13.1.1 Delta (and simplicial) complexes

- The **standard n-simplex** is $\Delta^n = \{(x_0, \ldots, x_n) \in \mathbb{R}^{n+1} \mid x_i \geqslant 0 \text{ and } \sum_i x_i = 1\}$.
- A **face** of a simplex with vertices $[x_0, \ldots, x_n]$ is any $[x_0, \ldots, \widehat{x_i}, \ldots, x_n]$.

Note that the vertices of any subsimplex are ordered *according to their order in the simplex that contains them*.

- The **boundary** $\partial\Delta^n$ of an n-simplex is the union of all of the faces of Δ^n.
- The **open simplex** $\mathring{\Delta}^n$ is the interior of Δ^n, i.e. $\Delta^n \setminus \partial\Delta^n$.
- A **delta-complex structure on a space** X is a collection of maps $\sigma_\alpha\colon \Delta^{n(\alpha)} \to X$ such that
  1. $\sigma_\alpha|_{\mathring{\Delta}^n}$ is injective;
  2. each point $x \in X$ is in the image of exactly one $\sigma_\alpha|_{\mathring{\Delta}^n}$;
  3. for each face f of Δ^n, the restriction $\sigma_\alpha|_f$ is one of the maps $\sigma_\beta\colon \Delta^{n-1} \to X$,

  and we define a topology by saying that $A \subset X$ is open iff $\sigma_\alpha^{-1}(A)$ is open in Δ^n for all σ_α.
- A **simplicial complex** is a delta complex whose simplices are *uniquely determined by their vertices*.

13.1.2 Simplicial homology

Let X be a Δ-complex.

- The **group of simplicial n-chains** $\Delta_n(X)$ of X is the free abelian group on the set of open n-simplices e^n_α of X.
- The **characteristic map** of an open n-simplex e^n_α is the σ_α such that $\sigma_\alpha(\mathring{\Delta}^n) = e^n_\alpha$.
- We can write n-chains as either $\sum_\alpha n_\alpha e^n_\alpha$ or $\sum_\alpha n_\alpha \sigma_\alpha$ for $n_\alpha \in \mathbb{N}$, where σ_α is the characteristic map of e^n_α.
- The **nth simplicial boundary map** $\partial_n\colon \Delta_n(X) \to \Delta_{n-1}(X)$ is the homomorphism defined by $\partial_n(\sigma_\alpha) = \sum_i (-1)^i \sigma_\alpha|_{[x_0, \ldots, \widehat{x_i}, \ldots, x_n]}$.
- The elements of $\ker \partial_n$ are called **n-cycles** and the elements of $\operatorname{im} \partial_{n+1}$ are called **n-boundaries**.
- The **nth simplicial homology group** $H^\Delta_n(X)$ of X is the nth homology of the chain complex $(\partial_\bullet(X), \partial_\bullet)$, i.e. $H^\Delta_n(X) = \ker(\partial_n)/\operatorname{im}(\partial_{n+1})$.

13.1.3 Singular homology

- A **singular n-simplex** in a space X is any (continuous) map $\sigma\colon \Delta^n \to X$.
- The **group of singular n-chains** $C_n(X)$ of X is the free abelian group on the set of singular n-simplices of X.
- The n**th singular homology group** $H_n(X)$ of X is the nth homology of the chain complex $(C_\bullet(X), \partial_\bullet)$, where $\partial_n\colon H_n(X) \to H_{n-1}(X)$ is the same as in the simplicial case.

13.1.4 Reduced homology

If X is a *non-empty* space then the n**th reduced homology group** $\widetilde{H}_n(X)$ is the nth homology group of the augmented chain complex $C_\bullet(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$, where $\mathbb{Z} = C_{-1}(X)$ is in degree -1 and $\varepsilon\left(\sum_i n_i\sigma_i\right) = \sigma_i n_i$.

- $H_0(X) \cong \widetilde{H}_0(X) \oplus \mathbb{Z}$.
- $H_n(X) \cong \widetilde{H}_n(X)$ for $n \geqslant 1$.

13.2 Cellular

13.2.1 CW complexes

- The **standard n-cell** is $D^n = \{x \in \mathbb{R}^n : \|x\| < 1\}$.
- A **CW complex** (or **cell complex**) is a space constructed by the following inductive method:
  1. Let $\{e_\alpha^n\}$ be a *non-empty* collection of n-cells for $n = 0, 1, 2, \ldots, k$ (where k might be in $\mathbb{N}$ or might be ∞).
  2. Define $X^0 = \{e_\alpha^0\}$.
  3. Define the n**-skeleton** X^n by $X^n = \left(X^{n-1} \sqcup_\alpha \overline{D_\alpha^n}\right)/\{x \sim \varphi_\alpha^n(x)\}$, i.e. by attaching each e_α^n to X^{n-1} via a map $\varphi_\alpha^n\colon S^{n-1} \to X^{n-1}$, where the map tells us how the boundary of the closed n-disc gets mapped into X^{n-1}.
  4. If $k \in \mathbb{N}$ then set $X = X^k$; if $k = \infty$ then set $X = \bigcup_n X^n$ and endow X with the **weak topology**: $U \subset X$ is open iff $U \cap X^n$ is open for all n.

13.2.2 Cellular homology

- The **group of cellular n-chains** $C_n^{\mathsf{CW}}(X)$ of X is defined by $C_n^{\mathsf{CW}}(X) = H_n(X^n, X^{n-1})$ and can be thought of as the free abelian group on the set of n-cells of X.
- The **cellular boundary map** $\partial_n^{\mathsf{CW}}\colon C_n^{\mathsf{CW}}(X) \to C_{n-1}^{\mathsf{CW}}(X)$ is induced by the long exact sequences for (X^k, X^{k-1}).
- The n**th cellular homology group** $H_n^{\mathsf{CW}}(X)$ of X is the nth homology of the complex $(C_\bullet^{\mathsf{CW}}, \partial_\bullet^{\mathsf{CW}})$.

There is a formula for the cellular boundary maps, constructed as follows. Let X be a CW complex, e_α^n an n-cell, and e_β^{n-1} an $n-1$-cell. Introduce the following maps:

- the attaching map φ_α^n;
- the quotient map $q^{n-1}\colon X^{n-1} \to X^{n-1}/X^{n-2}$;
- the quotient map $q_\beta^{n-1}\colon X^{n-1}/X^{n-2} \to S_\beta^{n-1}$ that collapses $X^{n-1} \setminus \{e_\beta^{n-1}\}$ to a point;
- the **attach-and-collapse map** $\chi_{\alpha\beta}^n\colon S_\alpha^{n-1} \to S_\beta^{n-1}$ given by $\chi_{\alpha\beta}^n = q_\beta^{n-1} q^{n-1} \varphi_\alpha^n$.

Then

1. $\partial_1^{\mathsf{CW}}(e_\alpha^1) = \partial_1(e_\alpha^1)$;
2. $\partial_n^{\mathsf{CW}}(e_\alpha^n) = \sum_\beta \left(\deg \chi_{\alpha\beta}^n\right) e_\beta^{n-1}$ for $n \geqslant 2$.

13.2.3 Examples

Find a CW-structure and then compute its homology for some common spaces, e.g. $\mathbb{RP}^2$ *and* K*.*

13.2.4 Moore spaces

Let G be a *finitely-generated* abelian group and $n \geqslant 1$ an integer.

- We define $M(\mathbb{Z}/m\mathbb{Z}, n)$ to be the CW complex given by attaching an $(n+1)$-cell to S^n by a map of degree m.
- We define the **Moore space** $M(G, n)$ as the CW complex $\bigvee_i M(G_i, n) \vee \bigvee_j S^n$, where G_i are the torsion summands of G and $j = 1, \ldots, z$, where z is the number of infinite cyclic summands of G.

1. $M(G, n)$ *is simply-connected for* $n \geqslant 2$*.*
2. $H_n(M(G, n)) \cong G$ *and* $\widetilde{H}_i(M(G, n)) = 0$ *for* $i \neq n$*.*

13.3 Interaction between types of homology

If X *is a delta complex then* $H_n^\Delta(X) \cong H_n(X)$*.*
If X *is a CW complex then* $H_n^\Delta(X) \cong H_n(X) \cong H_n^{CW}(X)$*.*
(The equivalent statements for relative homology are also true)

13.4 Homology and homotopy

- A map $f\colon X \to Y$ induces a chain map $f_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$ defined by composition, i.e. $f_n(\sigma) = f\sigma\colon \Delta^n \to Y$.
- A chain map $f_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$ induces a homomorphism $f_*\colon H_\bullet(X) \to H_\bullet(Y)$, i.e. homomorphisms $(f_*)_n\colon H_n(X) \to H_n(Y)$ for all n. If f_* is an *isomorphism* then we say that f is a **quasi-isomorphism**.

1. $(fg)_* = f_* g_*$
2. $\mathrm{id}_* = \mathrm{id}$
3. *If* $f, g\colon X \to Y$ *are homotopic then* $f_* = g_*$
4. *If* $f\colon X \to Y$ *is a homotopy equivalence then* f_* *is an isomorphism.*
5. *If* $f_\bullet, g_\bullet\colon C_\bullet(X) \to C_\bullet(Y)$ *are chain homotopic then* $f_* = g_*$

13.5 Relative

13.5.1 Relative homology

Let X be a space and $A \subset X$ a subspace.

- The **group of relative** n**-cycles** $C_n(X, A)$ is the quotient group $C_n(X)/C_n(A)$.
- The **relative boundary map** $\hat{\delta}_n\colon C_n(X, A) \to C_{n-1}(X, A)$ is the map induced by $\partial_n\colon C_n(X) \to C_{n-1}(X)$ (which is well-defined, since $\partial_n\colon C_n(A) \to C_{n-1}(A)$).
- The n**th relative homology group** $H_n(X, A)$ is the nth homology of the chain complex $(C_\bullet(X, A), \hat{\delta}_\bullet)$.

If $f, g\colon (X, A) \to (Y, B)$ *are homotopic through maps of pairs* $(X, A) \to (Y, B)$ *then* $f_* = g_*\colon H_\bullet(X, A) \to H_\bullet(Y, B)$*.*

For any pair (X, A) *with inclusion maps* $\iota\colon A \hookrightarrow X$ *and* $\iota'\colon (X, \varnothing) \hookrightarrow (X, A)$ *there is a long exact sequence*

$$\ldots \to H_n(A) \xrightarrow{\iota_*} H_n(X) \xrightarrow{\iota'_*} H_n(X, A) \xrightarrow{\partial_n} H_{n-1}(A) \xrightarrow{\iota_*} H_{n-1}(X) \xrightarrow{\iota'_*} H_{n-1}(X, A) \to \ldots$$

13.5.2 Excision

Excision theorem: (can be phrased in either of the two following equivalent ways)

1. *Let $Z \subset A \subset X$ be such that $\overline{Z} \subset \mathring{A}$. Then the inclusion $\iota\colon (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism ι_* on homology groups, i.e. $H_n(X \setminus Z, A \setminus Z) \cong H_n(X, A)$.*
2. *Let $A, B \subset X$ be such that $X \subset \mathring{A} \cup \mathring{B}$. Then the inclusion $\iota\colon (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism ι_* on homology groups, i.e. $H_n(B, A \cap B) \cong H_n(X, A)$.*

The **local homology groups of a space** X **at a point** x are the groups $H_n(X, X \setminus \{x\})$, which (assuming that points are closed in X) excision tells us are isomorphic to the groups $H_n(U, U \setminus \{x\})$ for any open set U containing x.

13.5.3 Good pairs

- A pair of spaces (X, A), where $A \subset X$, is called a **good pair** if
  1. A is closed;
  2. A is non-empty;
  3. A is a deformation retract of some neighbourhood in A.
- *If X is a CW complex and A a non-empty subcomplex then (X, A) is a good pair.*
- *If (X, A) is a good pair with inclusion map $\iota\colon A \hookrightarrow X$ and quotient map $\pi\colon X \twoheadrightarrow X/A$ then*
  1. *the induced quotient map $\hat{\pi}\colon (X, A) \to (X/A, A/A)$ induces an isomorphism $\hat{\pi}_*$ on homology groups, i.e. $H_n(X, A) \cong H_n(X/A, A/A)$;*
  2. $H_n(X/A, A/A) \cong \widetilde{H}_n(X/A)$.

  Thus the long exact sequence becomes

$$\ldots \to \widetilde{H}_n(A) \xrightarrow{\iota_*} \widetilde{H}_n(X) \xrightarrow{\pi_*} \widetilde{H}_n(X/A) \xrightarrow{\partial_n} \widetilde{H}_{n-1}(A) \xrightarrow{\iota_*} \widetilde{H}_{n-1}(X) \xrightarrow{\pi_*} \widetilde{H}_{n-1}(X/A) \to \ldots$$

13.5.4 Examples

1. *Calculate $H_n(D^n, \delta D^n)$*
2. *Do Example 2.23*
3. *Prove the Brouwer fixed point theorem*
4. *Prove Theorem 2.26*

13.6 Degree

13.6.1 Degree and local degree

Note that $H_n(S^n) \cong \mathbb{Z}$ for $n \geqslant 1$. Thus any map $f\colon S^n \to S^n$ induces a map $f_*\colon \mathbb{Z} \to \mathbb{Z}$ of the form $f_*(n) = d \cdot n$ for some $d \in \mathbb{Z}$ that depends only on f. We call this d the **degree of** f.

1. $\deg \mathrm{id} = 1$.
2. *If f is not surjective then* $\deg f = 0$.
   Pick $x \in S^n \setminus f(S^n)$ and factor f as $S^n \to S^n \setminus \{x\} \to S^n$. Since $S^n \setminus \{x\}$ is contractible it has zero nth homology for $n \geqslant 1$. So f factors through zero.
3. *If $f \simeq g$ then* $\deg f = \deg g$ *(the converse happens to be true as well, though for much less trivial reasons).*
4. $\deg fg = \deg f \deg g$.

5. *If f is a reflection of S^n (i.e. fixes points in a subsphere S^{n-1} and interchanges the two complementary hemispheres) then* $\deg f = -1$.
6. *The antipodal map* $-\mathrm{id}\colon x \mapsto -x$ *has degree* $(-1)^{n+1}$.

   This follows from the fact that it is a composition of $n+1$ reflections, each of which flips the sign.
7. *If f has no fixed points then* $\deg f = (-1)^{n+1}$.

   Pick x such that $f(x) \neq x$. Then the line segment $(1-t)f(x) - tx$ from $f(x)$ to $-x$ doesn't pass through the origin. Thus $f_t(x) = \frac{(1-t)f(x)-tx}{\|(1-t)f(x)-tx\|}$ defines a homotopy from f to $-\mathrm{id}$.

If $f\colon S^n \to S^n$ has the property that *some* point $y \in S^n$ has finitely many preimages $x_1, \dots, x_m$ then $(f_*)_n\colon H_n(U_i, U_i \setminus \{x_i\}) \to H_n(V, V \setminus \{y\})$ can be thought of as $f_*\colon \mathbb{Z} \to \mathbb{Z}$ (by using excision on a neighbourhood U_i of x_i with $f(U_i) \subseteq V$ for some neighbourhood V of y, and the long exact sequence of pairs), and so we can define the **local degree** $\deg f|_{x_i}$ **of** f **at** x_i in the same way as we defined the degree.

1. $\deg f = \sum_i \deg f|_{x_i}$
2. *if f maps each U_i homeomorphically to V then* $\deg f|_{x_i} = \pm 1$ *for all i.*

13.6.2 Example

example 2.31

13.7 Mayer-Vietoris

13.7.1 The derivation

Let $A, B \subset X$ be such that $X = \mathring{A} \cup \mathring{B}$. Define $C_n(A+B) \leqslant C_n(X)$ to be the subgroup consisting of n-chains that are sums of n-chains in A and n-chains in B. It can be shown that $C_n(A+B) \cong C_n(X)$, and so $(C_\bullet(A+B), \partial_\bullet)$ is a chain complex. We then apply the zig-zag lemma to the complex of short exact sequences coming from

$$\begin{array}{ccccc} 0 \to C_n(A \cap B) & \xrightarrow{\varphi} & C_n(A) \oplus C_n(B) & \xrightarrow{\psi} & C_n(A+B) \to 0 \\ x & \mapsto & (x, -x) & & \\ & & (y, z) & \mapsto & y + z. \end{array}$$

13.7.2 The sequence

If $A, B \subset X$ are such that $X = \mathring{A} \cup \mathring{B}$ then we have the long exact sequence

$$\dots \to H_n(A \cap B) \xrightarrow{\Phi} H_n(A) \oplus H_n(B) \xrightarrow{\Psi} H_n(X) \xrightarrow{\partial_n} H_{n-1}(A \cap B) \xrightarrow{\Phi} H_{n-1}(A) \oplus H_{n-1}(B) \xrightarrow{\Psi} H_{n-1}(X) \to \dots$$

13.7.3 The relative sequence

Let $Y \subset X$. If we have subspaces $C \subset A \subset X$ and $D \subset B \subset X$ such that

1. $X = \mathring{A} \cup \mathring{B}$
2. $Y = \mathring{C} \cup \mathring{D}$

then we have the long exact sequence

$$\dots \to H_n(A \cap B, C \cap D) \xrightarrow{\Phi} H_n(A, C) \oplus H_n(B, D) \xrightarrow{\Psi} H_n(X, Y) \xrightarrow{\partial_n} H_{n-1}(A \cap B, C \cap D) \xrightarrow{\Phi} H_{n-1}(A, C) \oplus H_{n-1}(B, D) \xrightarrow{\Psi} H_{n-1}(X, Y) \to \dots$$

13.7.4 Examples

Find and work through some examples.

13.8 Non-integer coefficients

Recall the following useful properties of Tor:

1. $\mathrm{Tor}(A, B) \cong \mathrm{Tor}(B, A)$;
2. $\mathrm{Tor}(\bigoplus_i A_i, B) \cong \bigoplus_i \mathrm{Tor}(A_i, B)$;
3. $\mathrm{Tor}(F, B) = \mathrm{Tor}(T_F, B) = 0$ for any free F or torsion-free T_F;
4. $\mathrm{Tor}(A, B) = \mathrm{Tor}(\mathrm{tors}(A), B)$.

Let X be a space, $A \subset X$ a subspace, and G an arbitrary *abelian* group.

- The n**-chain group in** X **with coefficients in** G is $C_n(X; G)$, defined as the free abelian group consisting of finite formal sums $\sum_i g_i\sigma_i$, called **chains**, where $g_i \in G$. Equivalently, $C_n(X; G) = \bigoplus_{\sigma \subset X} G$, where the sum is taken over all singular n-simplices in X.
- The **relative** n**-chain group** $C_n(X, A; G)$ is the quotient group $C_n(X; G)/C_n(A; G)$. This is also a direct sum: $C_n(X, A; G) = \bigoplus_{\sigma \subset X, \sigma \not\subset A} G$, where the sum is taken over all singular n-simplices is X but not in A.

1. *$C_n(X, A; G) \cong C_n(X, A) \otimes G$ naturally, via the map $\sum_i g_i\sigma_i \mapsto \sum_i(\sigma_i \otimes g_i)$.*
2. *Under the above isomorphism, the boundary map $\partial_n\colon C_n(X, A; G) \to C_{n-1}(X, A; G)$ becomes the map $\partial_n \otimes \mathrm{id}\colon C_n(X, A) \otimes G \to C_{n-1}(X, A) \otimes G$, where ∂_n here is the usual boundary map for coefficients in $\mathbb{Z}$.*
3. **Universal coefficient theorem for homology:** *If $C_\bullet$ is a chain complex of free abelian groups and G an arbitrary abelian group, then there are natural (in $C_\bullet$) short exact sequences*

$$0 \to H_n(C_\bullet) \otimes G \to H_n(C_\bullet; G) \to \mathrm{Tor}(H_{n-1}(C_\bullet), G) \to 0$$

   for all n. Further, these sequences split, though not *naturally.*
4. **Universal coefficient theorem for homology of a space:**

$$H_n(X; G) \cong (H_n(X) \otimes G) \oplus \mathrm{Tor}(H_{n-1}(X), G)$$

5. *If $f\colon S^n \to S^n$ has degree d then $f_*\colon H_n(S^n; G) \to H_n(S^n; G)$ is multiplication by d.*

13.8.1 Examples

Find and work through some examples.

13.9 Eilenberg and Steenrod axioms

- $\mathsf{Comp}_{\mathsf{CW}}$ is the category whose objects are CW complexes and who morphisms are maps of CW complexes.
- $\mathsf{Ab}_{\mathbb{N}}$ is the category whose objects are sequences $(A_0, A_1, A_2, \ldots)$ of abelian groups and whose morphisms are sequences $(\varphi_0, \varphi_1, \varphi_2, \ldots)$ of homomorphisms.

A **reduced homology theory on** $\mathsf{Comp}_{\mathsf{CW}}$ is a (covariant) functor $\widetilde{h}\colon \mathsf{Comp}_{\mathsf{CW}} \to \mathsf{Ab}_{\mathbb{N}}$ such that

1. if $f \simeq g\colon X \to Y$ then $\widetilde{h}(f)_n = \widetilde{h}(g)_n\colon \widetilde{h}(X)_n \to \widetilde{h}(Y)_n$;

2. for each CW pair (X, A) there are *natural* maps $\partial_n \colon \widetilde{h}(X/A)_n \to \widetilde{h}(A)_{n-1}$, called **boundary maps**, that give the long exact sequence

$$\ldots \to \widetilde{h}(A)_n \xrightarrow{\iota_*} \widetilde{h}(X)_n \xrightarrow{\pi_*} \widetilde{h}(X/A)_n \xrightarrow{\partial_n} \widetilde{h}(A)_{n-1} \xrightarrow{\iota_*} \widetilde{h}(X)_{n-1} \xrightarrow{\pi_*} \widetilde{h}(X/A)_{n-1} \to \ldots$$

   where $\iota \colon A \hookrightarrow X$ is the inclusion map and $\pi \colon X \twoheadrightarrow X/A$ is the quotient map;
3. the map $\bigoplus_\alpha \widetilde{h}(\iota_\alpha)_n \colon \bigoplus_\alpha \widetilde{h}(X_\alpha)_n \to \widetilde{h}(X)_n$ is an isomorphism for each n, where $X = \bigvee_\alpha X_\alpha$ is the wedge sum and $\iota_\alpha \colon X_\alpha \hookrightarrow X$ are the inclusion maps.

13.10 Useful facts and methods of calculation

- $H_0(X)$ is a direct sum of k copies of $\mathbb{Z}$, where k is the number of path-connected components of X.
- If $(X_\alpha, \{x_\alpha\})$ is a collection of good pairs then the inclusions $\iota_\alpha \colon X_\alpha \hookrightarrow \bigvee_\alpha X_\alpha$ (where the wedge sum is taken with basepoints being the x_α) induce an isomorphism $\bigoplus_\alpha (\iota_\alpha)_*$ on *reduced* homology groups, i.e. $\left(\bigoplus_\alpha \widetilde{H}_n(X_\alpha)\right) \cong \widetilde{H}_n(\bigvee_\alpha X_\alpha)$.
- Use long exact sequences, e.g. Corollary 2.14.
- Use the explicit delta/CW complex structure.

Come up with some flowchart of ways to calculate homology.

13.11 Homology of common spaces

Compute the homology of the genus-n orientable surface, the genus-n non-orientable surface, $\mathbb{R}^n \setminus \{p_1, \ldots, p_k\}$, and any other common spaces.

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14 Cohomology

14.1 General definition

Let $(C_\bullet, \partial_\bullet)$ be a chain complex of free abelian groups and G an arbitrary *abelian* group.

- The **cochain group** C_n^* is defined by $C_n^* = \mathrm{Hom}(C_n, G)$.
- The **coboundary map** $\delta_n = \partial_{n+1}^* \colon C_n^* \to C_{n+1}^*$ is defined by precomposition by ∂_{n+1}.
- The **cohomology group** $H^n(C_\bullet; G)$ **of** $C_\bullet$ **with coefficients in** G is defined as the nth cohomology of the cochain complex $(C_\bullet^*, \delta_\bullet)$, i.e. $\ker \delta_n / \operatorname{im} \delta_{n-1}$.

14.2 Universal coefficients

Recall the following useful properties of Ext:

1. $\mathrm{Ext}(H \oplus H', G) \cong \mathrm{Ext}(H, G) \oplus \mathrm{Ext}(H', G)$;
2. $\mathrm{Ext}(F, G) = 0$ for any free abelian group F;
3. $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, G) \cong G/nG$.

Let $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ be chain complexes of free abelian *groups, and G an arbitrary* abelian *group. Then the following facts hold:*

1. **Universal coefficient theorem for cohomology:** *The cohomology groups $H^n(C_\bullet; G)$ are determined by the* split *exact sequence*

$$0 \to \operatorname{Ext}(H_{n-1}(C_\bullet), G) \to H^n(C_\bullet; G) \to \operatorname{Hom}(H_n(C_\bullet), G) \to 0.$$

   Further, this sequence is natural (and splits naturally) in $C_\bullet$.
2. **Universal coefficient theorem for cohomology of a space:**

$$H^n(X; G) \cong \operatorname{Ext}(H_{n-1}(X), G) \oplus \operatorname{Hom}(H_n(X), G)$$

3. *If the homology groups $H_n(C_\bullet)$ and $H_{n-1}(C_\bullet)$ are finitely generated then*

$$H^n(C_\bullet; \mathbb{Z}) \cong \frac{H_n(C_\bullet)}{\operatorname{tors}\big(H_n(C_\bullet)\big)} \oplus \operatorname{tors}\big(H_{n-1}(C_\bullet)\big)$$

   where $\operatorname{tors}(A)$ is the torsion subgroup of an abelian group A.
4. *If $f_\bullet \colon C_\bullet \to C'_\bullet$ is a quasi-isomorphism then it also induces an isomorphism on cohomology groups.*

14.3 Cohomology of a space

14.3.1 Definitions

Let X be some space and G an arbitrary abelian group.

- The **group of singular n-cochains $C^n(X; G)$ with coefficients in** G is the dual group $\operatorname{Hom}(C_n(X), G)$ to the singular n-chain group $C_n(X)$, and is thus equivalent to the group of functions from singular n-simplices to G.
- The **coboundary map** $\delta_n \colon C^n(X) \to C^{n+1}(X)$ is the dual ∂^*_{n+1} defined by precomposition by the singular boundary map $\partial_{n+1} \colon C_{n+1}(X) \to C_n(X)$.
- The elements of $\ker \delta_n$ are called n-**cocyles** and the elements of $\operatorname{im} \delta_{n-1}$ are called n-**coboundaries**.
- The n**th singular cohomology group of** X **with coefficients in** G is defined as the nth cohomology of the cochain complex $(C^\bullet, \delta_\bullet)$, i.e. $\ker \delta_n / \operatorname{im} \delta_{n-1}$.

14.3.2 Reduced cohomology

The **reduced cohomology groups** $\widetilde{H}^n(X; G)$ are defined by dualising the augmented chain complex $C_\bullet(X) \xrightarrow{\varepsilon} \mathbb{Z} \to 0$ used to define reduced homology, where $\mathbb{Z} = C_{-1}(X)$ is in degree 1 and $\varepsilon\left(\sum_i n_i \sigma_i\right) = \sum_i n_i$.

- $H^0(X; G)$ is the group of functions $X \to G$ that are constant on path-components; $\widetilde{H}^0(X; G)$ is the group of functions $X \to G$ that are constant on path components *modulo the functions that are constant on all of X*.
- $H^n(X; G) \cong \widetilde{H}^n(X; G)$ for $n \geqslant 1$.

14.3.3 Relative cohomology and the long exact sequence of a pair

Let $A \subset X$ some subspace with inclusion map $\iota \colon A \hookrightarrow X$ and quotient map $\pi \colon X \twoheadrightarrow X/A$.

- *The dual*

$$0 \leftarrow C^n(A; G) \xleftarrow{\iota^*} C^n(X; G) \xleftarrow{\pi^*} C^n(X, A) \leftarrow 0$$

  of the short exact sequence $0 \to C_n(A) \xrightarrow{\iota} C_n(X) \xrightarrow{\pi} C_n(X, A) \to 0$ *is exact.*
  We know that $Hom(-, G)$ is left-exact, but *contravariant*, and so the right-hand side of the dual sequence is exact. To see that ι^* is surjective note that it is simply the restriction of cochains on X to cochains on A, but given some cochain on A we can extend it to a cochain on X by defining to to be zero on $X \setminus A$, and this will map back to the original cochain under ι^*. To see that $\ker \iota^* = \operatorname{im} \pi^*$ note that the kernel consists of n-cochains that are zero on singular n-simplices in A, which are exactly the homomorphisms $C_n(X, A) = C_n(X)/C_n(A) \to G$. Thus $\ker \iota^* = \operatorname{Hom}(C_n(X, A), G) = C^n(X, A; G)$.

- *$C^n(X, A; G)$ is the group of functions from singular n-simplices in X to G that vanish on simplices in A.*

  This is since $C_n(X) \cong C_n(A) \oplus C_n(X, A)$ where the two groups on the right are disjoint, and one consists of simplices with image in A and the other consists of simplices with image *not* in A.
- The **relative coboundary map** $\delta_n \colon C^n(X, A; G) \to C^{n+1}(X, A; G)$ is defined as the restriction of $\delta_n \colon C^n(X; G) \to C^{n+1}(X; G)$ using the above interpretation.
- The n**th relative cohomology group** $H^n(X, A; G)$ is defined as the nth cohomology of the cochain complex $(C^\bullet, \delta_\bullet)$, i.e. $\ker \delta_n / \operatorname{im} \delta_{n-1}$.
- *We have the long exact sequence*

$$\ldots \leftarrow H^{n+1}(A; G) \xleftarrow{\iota^*} H^{n+1}(X; G) \xleftarrow{\pi^*} H^{n+1}(X, A; G) \xleftarrow{\delta_n} H^n(A; G) \xleftarrow{\iota^*} H^n(X; G) \xleftarrow{\pi^*} H^n(X, A; G) \leftarrow \ldots$$

- *The following diagram (with the connecting homomorphisms ∂_n and δ_n from the long exact sequences of (co)homology) commutes:*

$$\begin{array}{ccc} H^n(A; G) & \xrightarrow{\delta_n} & H^{n+1}(X, A; G) \\ \downarrow h & & \downarrow h \\ \operatorname{Hom}(H_n(A), G) & \xrightarrow{\partial_n^*} & \operatorname{Hom}(H_{n+1}(X, A), G) \end{array}$$

  where h is simply restriction to cycles in $C_n(A)$ or to relative cycles in $C_n(X, A)$.

Note that the universal coefficient theorem for relative cohomology also holds.

14.3.4 Induced homomorphisms and homotopy invariance

Let X and Y be spaces

- A map $f \colon X \to Y$ induces a chain map $f^\bullet \colon C^\bullet(Y) \to C^\bullet(X)$.
- A chain map $f^\bullet \colon C^\bullet(Y) \to C^\bullet(X)$ induces a homomorphism $f^* \colon H^n(Y; G) \to H^n(X; G)$, i.e. homomorphisms $(f^*)^n \colon H^n(Y) \to H^n(X)$ for all n.

1. $(fg)^* = g^* f^*$
2. $\mathrm{id}^* = \mathrm{id}$
3. *If $f, g \colon X \to Y$ are homotopic then $f^* = g^*$*
4. *If $f \colon X \to Y$ is a homotopy equivalence then f^* is an isomorphism.*
5. *If $f^\bullet, g^\bullet \colon C^\bullet(Y) \to C^\bullet$ are chain homotopic then $f^* = g^*$*

The equivalent statements for relative cohomology also hold.

14.3.5 Excision

Excision theorem: (can be phrased in either of the two following equivalent ways)

1. *Let $Z \subset A \subset X$ be such that $\overline{Z} \subset \mathring{A}$. Then the inclusion $\iota \colon (X \setminus Z, A \setminus Z) \hookrightarrow (X, A)$ induces an isomorphism ι^* on cohomology groups, i.e. $H^n(X, A) \cong H^n(X \setminus Z, A \setminus Z)$.*
2. *Let $A, B \subset X$ be such that $X \subset \mathring{A} \cup \mathring{B}$. Then the inclusion $\iota \colon (B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism ι^* on homology groups, i.e. $H^n(X, A) \cong H^n(B, A \cap B)$.*

14.3.6 Mayer-Vietoris

If $A, B \subset X$ are such that $X = \mathring{A} \cup \mathring{B}$ then we have the long exact sequence

$$\ldots \leftarrow H^{n+1}(A \cap B; G) \xleftarrow{\Phi} H^{n+1}(A; G) \oplus H^{n+1}(B; G) \xleftarrow{\Psi} H^{n+1}(X; G) \xleftarrow{\delta_n} H^n(A \cap B; G) \xleftarrow{\Phi} H^n(A; G) \oplus H^n(B; G) \xleftarrow{\Psi} H^n(X; G) \leftarrow \ldots$$

14.3.7 Eilenberg and Steenrod axioms

These are exactly dual to the axioms for a reduced homology theory, but note that $\operatorname{Hom}\left(\bigoplus_\alpha A_\alpha, B\right) \cong \prod_\alpha \operatorname{Hom}(A_\alpha, B)$, and so the wedge axiom will require that $\prod_\alpha \widetilde{h}(\iota_\alpha)^n$ be an isomorphism.

14.4 Cup product

14.4.1 Definition

Given a space X, let R be some ring (usually one of $\mathbb{Z}, \mathbb{Z}/m\mathbb{Z}, \mathbb{Q}$, though not necessarily commutative), and let $\varphi \in C^k(X;R), \psi \in C^\ell(X;R)$ be cochains.

- The **cup product** $\varphi \smile \psi \in C^{k+\ell}(X;R)$ is the cochain whose value on a singular simplex $\sigma\colon \Delta^{k+\ell} \to X$ is given by

  $$(\varphi \smile \psi)(\sigma) = \varphi(\sigma|_{[x_0,\ldots,x_k]}) \cdot \psi(\sigma|_{[x_k,\ldots,x_{k+\ell}]})$$

  where the multiplication $\cdot$ on the right-hand side is in R.
- *The following relation, where δ is the coboundary map, holds:*

  $$\delta_{k+\ell}(\varphi \smile \psi) = (\delta_k\varphi \smile \psi) + (-1)^k(\varphi \smile \delta_\ell\psi).$$

- *The cup product of two cocycles is again a cocycle; the cup product of a cocycle and a coboundary (in either order) is a coboundary.*
- Using the previous property, there is an **induced cup product** on homology:

  $$H^k(X;R) \times H^\ell(X;R) \xrightarrow{\smile} H^{k+\ell}(X;R).$$

  This is associative and distributive (since it is both at the level of cochains). Further, if R has identity 1_R then the induced cup product has identity $(\sigma^0 \mapsto 1_R) \in H^1(X;R)$.

Note that we can define the cup product for simplicial homology in the same way, so the canonical isomorphism between singular and simplicial homology respects the cup product.

14.4.2 The relative case

*Let $A, B \subset X$ be open subsets (or subcomplexes, if X is a CW complex). Using the same definition as in the absolute case, we have the **relative cup product** $\smile$ which induces*

$$H^k(X,A;R) \times H^\ell(X,B;R) \xrightarrow{\smile} H^{k+\ell}(X, A\cup B;R).$$

This follows from the fact that the absolute cup product restricts to a cup product $C^k(X,A;R) \times C^\ell(X,B;R) \to C^{k+\ell}(X, A+B;R)$, where $C^n(X,A+B;R) \leqslant C^n(X;R)$ is the subgroup consisting of cochains vanishing on sums of chains in A and chains in B. If A and B are open (or subcomplexes, and X is a CW complex) then the inclusions $C^n(X, A\cup B;R) \hookrightarrow C^n(X,A+B;R)$ induce isomorphisms on cohomology via the five lemma and excision.

14.4.3 Properties

- *Let $f\colon X \to Y$ be a map of spaces. Then the induced maps $f^*\colon H^n(Y;R) \to H^n(X;R)$ satisfy $f^*(\alpha \smile \beta) = f^*(\alpha) \smile f^*(\beta)$.*
- *If R is commutative then $\alpha \smile \beta = (-1)^{k\ell}\beta \smile \alpha$ for all $\alpha \in H^k(X;R), \beta \in H^\ell(X;R)$.*

The equivalent statements for relative cohomology also hold.

14.4.4 The cohomology ring

- The **cohomology ring** $H^*(X, A; R)$ is the *graded* ring $\bigoplus_n H^n(X, A; R)$. It can naturally be given an R-algebra structure.
- The **dimension** $|\alpha|$ **of an element** $a \in H^*(X, A; R)$ is the integer $k \in \mathbb{N}$ such that $a \in H^k(X, A; R)$. This means that (when R is commutative) we can write the *graded-commutativity property* as $a \smile b = (-1)^{|a||b|} b \smile a$.

14.4.5 Examples

Work through Examples 3.7–3.10 and 3.12–3.14.

14.5 Künneth formula

14.5.1 Cross product and a Künneth formula

Let X, Y be spaces and R a *commutative* ring. Write $\pi_X : X \times Y \to X$ and $\pi_Y : X \times Y \to Y$ for the projection maps of the product of spaces.

- The **cross product** (or **external cup product**) $\times$ is the R-module homomorphism

$$H^*(X; R) \otimes_R H^*(Y; R) \xrightarrow{\times} H^*(X \times Y; R)$$

  where we define $a \times b = \pi_X^*(a) \smile \pi_Y^*(b)$.
- If we define multiplication on the tensor product by $(a \otimes b)(c \otimes d) = (-1)^{|b||c|} ac \otimes bd$ then the cross product becomes a ring homomorphism.
- **Künneth theorem:** *The cross product is an isomorphism of rings if X and Y are CW complexes and $H^n(Y; R)$ is a free, finitely-generated R-module for all n.*
- **Relative Künneth theorem:** *The relative cross product*

$$H^*(X, A; R) \otimes_R H^*(Y, B; R) \xrightarrow{\times} H^*(X \times Y, A \times Y \cup X \times B; R)$$

  is an isomorphism of rings if (X, A) and (Y, B) are CW pairs and $H^n(Y, B; R)$ is a free, finitely-generated R-module for all n.

14.5.2 Examples

Find and work through some simple examples.

Prove that $H^*(\mathbb{R}p^n; \mathbb{Z}/2\mathbb{Z}) \cong \mathbb{Z}/2\mathbb{Z}[\alpha](\alpha^{n+1})$ where $|\alpha| = 1$.

14.6 Useful facts and methods of calculation

Come up with some flowchart of ways to calculate cohomology. Include using duality.

14.7 Cohomology of common spaces

Compute the cohomology of the genus-n orientable surface, the genus-n non-orientable surface, $\mathbb{R}^n \setminus \{p_1, \ldots, p_k\}$, and any other common spaces. Work with coefficients in $\mathbb{Z}$ and $\mathbb{Z}/p\mathbb{Z}$.

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15 Duality

15.1 Manifolds

- An n-**manifold** is a *Hausdorff* space where every point has a neighbourhood homeomorphic to $\mathbb{R}^n$.
- A manifold is called **closed** if it is a compact topological space.

- *If M is an n-manifold then the local homology group $H_k(M, M \setminus \{x\}; \mathbb{Z}) = \delta_{kn}\mathbb{Z}$ for any $x \in M$, where δ_{ij} is the Kronecker delta.*

$$\begin{aligned} H_k(M, M \setminus \{x\}) &\cong H_k(\mathbb{R}^n, \mathbb{R}^n \setminus \{0\}) && \text{by excision} \\ &\cong \widetilde{H}_{k-1}(\mathbb{R}^n \setminus \{0\}) && \text{since } \mathbb{R}^n \text{ is contractible} \\ &\cong \widetilde{H}_{k-1}(S^{n-1}) && \text{since } \mathbb{R}^n \setminus \{0\} \simeq S^{n-1} \end{aligned}$$

Note: for $A \subset M$ we write $H_n(M \mid A)$ to mean $H_n(M, M \setminus A)$, and we abuse notation in writing $H_n(M \mid x)$ to mean $H_n(M \mid \{x\})$.

15.2 Orientability

Let M be an n-manifold and $x \in M$ a point in M.

- A **local orientation of** M **at** x is a choice of generator μ_x of the infinite cyclic group $H_n(M \mid x; \mathbb{Z}) \cong \mathbb{Z}$.
- An **orientation of** M is an assignment $x \mapsto \mu_x$ of a local orientation to each point $x \in M$ *such that* there exists some finite-radius open ball $B \subset M$ containing x, and some generator μ_B of $H_n(M \mid B) \cong H_n(\mathbb{R}^n \mid B)$ with μ_B mapping to μ_y under the natural map $H_n(M \mid B) \to H_n(M \mid y)$ for all $y \in B$.
- If we can find an orientation on M then we say that M is **orientable**.
- A **generator of a commutative ring** R **with identity** is $\mu \in R$ such that $R\mu = R$.
- We define R-**orientability** for any *commutative ring R with identity* similarly by assigning to each $x \in M$ a generator μ_x of $H_n(M \mid x; R) \cong R$ and requiring that it satisfies the same condition as when $R = \mathbb{Z}$.
- A **fundamental class for** M **with coefficients in** R is an element $[M] \in H_n(M; R)$ such that, for all $x \in M$, its image under $H_n(M; R) \to H_n(M \mid x; R)$ is a generator.
- An equivalent definition of R-**orientability** follows from the fact that *M is closed and R-orientable iff a fundamental class exists.*

1. *If M is a connected n-manifold then $H_i(M; R) = 0$ for all $i \geqslant n+1$.*
2. *Let M be a* closed *connected n-manifold. Then*
   (a) *for all $x \in M$, the map $H_n(M; R) \to H_n(M \mid x; R) \cong R$ is*
      (i) *an isomorphism if M is R-orientable;*
      (ii) *injective, with image $\{r \in R \mid 2r = 0\}$ if M is* not *R-orientable.*
   (b) *the torsion subgroup $\operatorname{tors}(H_{n-1}(M; \mathbb{Z}))$ is*
      (i) *trivial if M is $\mathbb{Z}$-orientable;*
      (ii) *$\mathbb{Z}/2\mathbb{Z}$ if M is* not *$\mathbb{Z}$-orientable.*
   (c) *M is R-orientable for*
      (i) *all R if M is $\mathbb{Z}$-orientable;*
      (ii) *all R containing a unit of order 2 if M is* not *$\mathbb{Z}$-orientable.*

15.2.1 Examples

Work through the paragraph about delta-complexes on p. 238 with an explicit example.

15.3 Poincaré

15.3.1 Cap product

Let X be a space and R a commutative ring with identity.

- The **cap product** is the R-bilinear map

$$C_k(X; R) \times C^\ell(X; R) \xrightarrow{\frown} C_{k-\ell}(X; R)$$

for $k \geqslant \ell$, defined by $\sigma \frown \varphi = \varphi(\sigma_{[x_0, \ldots, x_\ell]}) \cdot \sigma|_{[x_\ell, \ldots, x_k]}$.

- *The cap product satisfies $\partial_{k-\ell}(\sigma \frown \varphi) = (-1)^\ell(\partial_k\sigma \frown \varphi - \sigma \frown \delta_\ell\varphi)$.*
- The previous claim implies that the cap product of a cycle and a cocycle is a cycle, and that the cap product of either a boundary and a cocycle, or a cycle and a coboundary, is a boundary. This induces a **cap product on homology**:

$$H_k(X;R) \times H^\ell(X;R) \xrightarrow{\frown} H_{k-\ell}(X;R)$$

  and a **cap product on relative homology**:

$$H_k(X, A \cup B; R) \times H^\ell(X, A; R) \xrightarrow{\frown} H_{k-\ell}(X, B; R)$$

  when $A, B \subset X$ are open.

15.3.2 Closed manifolds

Poincaré duality: *Let M be a closed R-orientable n-manifold with fundamental class $[M] \in H_n(M;R)$. Then the map*

$$D : H^k(M;R) \to H_{n-k}(M;R)$$
$$\alpha \mapsto [M] \frown \alpha$$

is an isomorphism for all k.

15.3.3 Non-compact manifolds

- A space X is **locally compact** if each point has a compact neighbourhood.
- For a locally-compact space X we define **cohomology groups with compact support** $H^n_c(X;R) = H_n(C^\bullet_c(X);R)$, where

$$C^i_c(X) = \{\varphi \in C^i(X;R) \mid \exists K \subset X \text{ s.t. } K \text{ is compact and } \varphi|_{X \setminus K} = 0\}.$$

  Alternatively, we can define $H^i_c(X;R) = \operatorname{colim}_{\text{compact } K \subset X} H^i(X, X \setminus K; R)$.
- *Let M be an R-orientable n-manifold and $K \subset M$ a compact subspace. Then there exists a relative fundamental class $[K_M] \in H_n(M, M \setminus K)$.*
- **Poincaré duality (non-compact):** *Let M be an R-orientable n-manifold. Then the map $D_M \colon H^k_c(M;R) \to H_{n-k}(M;R)$ is an isomorphism for all k, where the map D_M is the colimit of the maps $H^i(M, M \setminus K; R) \to H_{n-k}(M;R)$ given by $\varphi \mapsto [K_M] \frown \varphi$.*

15.3.4 Cup and cap products

- *The cup and cap products satisfy $\psi(\sigma \frown \varphi) = (\varphi \smile \psi)(\sigma)$, where $\sigma \in C_{k+\ell}(X;R)$, $\varphi \in C^k(X;R)$, and $\psi \in C^\ell(X;R)$.*
- Let A and B be R-modules. A *bilinear* map $A \times B \to R$ is said to be **non-singular pairing** if the maps $A \to \operatorname{Hom}_{R-\mathsf{mod}}(B, R)$ and $B \to \operatorname{Hom}_{R-\mathsf{mod}}(A, R)$ are both isomorphisms, where the maps are induced by thinking of one of the variables as being fixed.
- *Let M be a closed R-orientable n-manifold. The the cup product induces a non-singular pairing, given by*

$$H^k(M;R) \times H^{n-k}(M;R) \to R$$
$$(\varphi, \psi) \mapsto (\varphi \smile \psi)[M].$$

  Proof: This follows from the fact that $\psi(\sigma \frown \varphi) = (\varphi \smile \psi)(\sigma)$ and Poincaré duality.
- *Let M be a closed $\mathbb{Z}$-orientable n-manifold, and $\varphi \in H^k(M;\mathbb{Z})$. Then φ is a generator of an infinite cyclic summand of $H^k(M;\mathbb{Z})$ iff there exists $\psi \in H^{n-k}(X;\mathbb{Z})$ such that $\varphi \smile \psi$ is a generator of $H^n(X;\mathbb{Z}) \cong \mathbb{Z}$.*
  Proof: φ generates a $\mathbb{Z}$-summand iff there exists a homomorphism $\vartheta \colon H^k(M;\mathbb{Z}) \to \mathbb{Z}$ such that $\vartheta(\varphi) = \pm 1$. Since the cup product pairing is non-singular, we know that every homomorphism is of the form $\vartheta = (- \smile \psi)[M]$ for some $\psi \in H^{n-k}(M;\mathbb{Z})$. So there exists ϑ satisfying $\vartheta(\varphi) = \pm 1$ iff there exists ψ such that $\varphi \smile \psi$ generates $H^n(M;\mathbb{Z})$.

15.3.5 Examples

Prove Corollary 3.37. | *Work through the example in Evie's notes on p. 52.*

15.4 Lefschetz

15.4.1 Lefschetz duality

- An n-**manifold with boundary** is a Hausdorff space in which each point has an open neighbourhood homeomorphic either to $\mathbb{R}^n$ or to $\mathbb{R}^n_+ = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geqslant 0\}$. The **boundary** ∂M of such a manifold M is the subspace of points whose open neighbourhoods are homeomorphic to $\mathbb{R}^n_+$, or, equivalently, that satisfy $H_n(M \mid x; \mathbb{Z}) = 0$.
- An n-manifold M with boundary ∂M is R**-orientable** if $M \setminus \partial M$ is an R-orientable n-manifold.
- *If M is a closed R-oriented manifold with boundary then there exists a fundamental class $[M] \in H_n(M, \partial M)$.*
- **Lefschetz duality:** *Let M be a closed R-oriented manifold with boundary. Then*
  1. $H^k(M, \partial M) \cong H_{n-k}(M)$;
  2. $H^k(M) \cong H_{n-k}(M, \partial M)$.

  Proof: First show that $H^k(M, \partial M; \mathbb{Z}) \cong H^k_c(M \setminus \partial M; \mathbb{Z})$ by using Poincaré duality for non-compact manifold. Then use the long exact sequence of $(M, \partial M)$ in homology and cohomology, relate them by the cap product, and use the five lemma.

15.4.2 Example

Work through the example in Evie's notes on p. 54.

15.5 Alexander

15.5.1 Alexander duality

- A space X is said to be **locally contractible** if it has a basis of **contractible** (i.e. homotopic to a point) open subsets.
- **Alexander duality:** *Let $K \subset S^n$ be a compact and locally contractible subspace. Then $\widetilde{H}_i(S^n \setminus K; \mathbb{Z}) \cong \widetilde{H}^{n-i-1}(K)$.*

  Proof:

$$\begin{aligned}
\widetilde{H}_i(S^n \setminus K) &\cong \widetilde{H}^{n-i}_C(S^n \setminus K) && \text{Poincaré duality} \\
&\cong \operatorname*{colim}_{\text{open } U \supset K} \widetilde{H}^{n-i}(S^n \setminus K, U \setminus K) && \text{by definition} \\
&\cong \operatorname*{colim}_{\text{open } U \supset K} \widetilde{H}^{n-i}(S^n, U) && \text{excision} \\
&\cong \operatorname*{colim}_{\text{open } U \supset K} \widetilde{H}^{n-i-1}(U) && \text{long exact sequence} \\
&\cong \widetilde{H}^{n-i-1}(K).
\end{aligned}$$

15.5.2 Example

Work through the example in Evie's notes on p. 55.

AT

16 The general Künneth formula

General Künneth formula: *Let X and Y be CW complexes, and R a* PID*. Then there are short exact sequences*

$$0 \to \bigoplus_i (H_i(X;R) \otimes_R H_{n-i}(Y;R)) \to H_n(X \times Y;R) \to \bigoplus_i \operatorname{Tor}^R(H_i(X;R), H_{n-i-1}(Y;R)) \to 0$$

natural in $X \times Y$, and that split (though not naturally), for all n.

AG 17 Affine varieties

Throughout, we assume that k is an algebraically closed field.

- A subset $X \subseteq k^n$ is an **affine variety** if it is of the form $\mathbb{V}(I)$ for some $I \lhd k[x_1, \ldots, x_n]$.
  - $\mathbb{V}(f) \subseteq k^n$ for a polynomial f is called a **hypersurface**.
  - $\mathbb{V}(f) \subseteq k^n$ for $f = a_1x_1 + \ldots + a_nx_n$ a linear form is called a **hyperplane**.
- The **Zariski topology on** k^n is the topology whose *closed* sets are the affine varieties, and we write $\mathbb{A}_k^n$ to mean k^n with the Zariski topology.
- The **Zariski topology on an affine variety** $X \subset k^n$ is given by the subspace topology: closed subsets of X are those of the form $X \cap \mathbb{V}(I)$ for some $I \lhd k[x_1, \ldots, x_n]$.
- An affine variety $X \subset \mathbb{A}_k^n$ is called **reducible** if it can be written as a non-trivial union of two non-trivial subvarieties: $X = X_1 \cup X_2$ with $X_1 \neq X_2$, and $X_1, X_2 \neq \varnothing$.
- A **morphism of affine spaces** $F\colon \mathbb{A}^m \to \mathbb{A}^n$ is given by n polynomial maps on $\mathbb{A}^m$, i.e. $x = (x_1, \ldots, x_n) \mapsto (f_1(x), \ldots, f_n(x))$, where $f_i \in k[x_1, \ldots, x_m]$.
- An **isomorphism** is a morphism with an inverse morphism.

AG 18 Projective varieties

- *If $F \in k[x_0, \ldots, x_n]$ is homogeneous of degree d then $F(\lambda x_0, \ldots, \lambda x_n) = \lambda^d F(x_0, \ldots, x_n)$.*
- An ideal $I \lhd k[x_0, \ldots, x_n]$ is said to be a **homogeneous ideal** if $I = (F_1, \ldots, F_r)$ with each F_i homogeneous of degree d_i.
- A subset $X \subset \mathbb{P}_k^n$ is a **projective variety** if it is of the form $\mathbb{V}(I)$ for some *homogeneous* ideal $I \lhd k[x_0, \ldots, x_n]$.
  - A **projective hypersurface** is $\mathbb{V}(F)$ for some *homogeneous* polynomial F.
  - If $L = a_0x_0 + \ldots + a_nx_n$ is a *non-zero* linear form then $\mathbb{V}(L)$ is called a **projective hyperplane**.
  - The i**th coordinate hyperplane** H_i is defined as $\mathbb{V}(x_i)$; its complement U_i is called the i**th coordinate chart**.
  - *If $X \subset \mathbb{P}^n$ is a projective variety then $\bigcup_{i=0}^n X \cap U_i$ is an open cover of X.*
- The **Zariski topology on** $\mathbb{P}_k^n$ is the topology whose closed subsets are the projective varieties.
- The **Zariski topology on a projective variety** $X \subset \mathbb{P}^n$ is given by the subspace topology: closed subsets of X are those of the form $X \cap \mathbb{V}(I)$ for some homogeneous $I \lhd k[x_1, \ldots, x_n]$.
- The **affine cone** $\widehat{X}$ over a projective variety $X = \mathbb{V}(I) \subset \mathbb{P}^n$ is the affine variety $\mathbb{V}(I) \subset \mathbb{A}^{n+1}$.
- Let $X \subseteq \mathbb{P}^m$ and $Y \subseteq \mathbb{P}^n$ be projective varieties. A map $F\colon X \to Y$ is a **morphism of projective varieties** if for all $x \in X$ there exists an open neighbourhood $U \subseteq X$ of p and homogeneous polynomials $f_0, \ldots, f_n \in k[x_0, \ldots, x_m]$, *all of the same degree*, such that $F|_U : U \to Y$ agrees with $[y] \mapsto [f_0(y) : \ldots : f_n(y)]$.
- An **isomorphism** is a morphism with an inverse morphism.
- We say that two projective varieties $X, Y \subseteq \mathbb{P}^n$ are **projectively equivalent** if they can be transformed into one another by a linear change of coordinates in $\mathbb{P}^n$, i.e. there exists a linear transformation $A \in \mathrm{GL}(n+1, k)$ which induces an isomorphism $A\colon X \to Y$ given by $[x_0 : \ldots : x_n] \mapsto [Ax_0 : \ldots : Ax_n]$.
- *Projective equivalence is a stronger relation that isomorphism.*
- We say that two projective varieties $X, Y \subseteq \mathbb{P}^n$ are **linearly equivalent** if they are isomorphic and the isomorphism is induced by a linear isomorphism $\mathbb{P}^n \cong \mathbb{P}^n$.

AG

19 Classical maps and embeddings

19.1 Veronese map

- Let V be a k-vector space. The **dth symmetric product** $\mathrm{Sym}^d(V)$ of V is the vector space given by the quotient of $V^{\otimes d} = \bigotimes_{i=1}^d V$ by the ideal of relations defined by the action of the symmetric group, i.e. identify tensors if they differ by transpositions: $v \otimes w \sim w \otimes v$. If we have a basis $\{v_i\}$ for V then the dth symmetric product is the vector space generated by the symbols $v_1^{\otimes i_1} \otimes \ldots \otimes v_n^{\otimes i_n}$ such that $\sum_j i_j = d$.
- *There are $\binom{n+d}{d}$ monomials of degree d in $n+1$ variables.*
- The **degree d Veronese map** is the map

$$v_d \colon \mathbb{P}^n \to \mathbb{P}^{\binom{n+d}{d}-1}$$
$$[x_0 : \ldots : x_n] \mapsto [\ldots : x^I : \ldots]$$

  where we run over all $I = (i_0, \ldots, i_n) \in \mathbb{N}^{n+1}$ such that $\sum_j i_j = d$, and define $x^I = x_0^{i_0} \ldots x_n^{i_n}$. **Equivalently** we can define it as

$$v_d \colon \mathbb{P}(\mathbb{A}^{n+1}) \to \mathbb{P}(\mathrm{Sym}^d \mathbb{A}^{n+1})$$
$$[x] \mapsto [x^d].$$

- The image of $v_d \colon \mathbb{P}^1 \hookrightarrow \mathbb{P}^d$ is called the **rational normal curve of degree** d.
- *There is an isomorphism*

$$\mathbb{P}^n \cong \mathrm{Im}\,(v_d) = \mathbb{V}\big(\{z_I z_J - z_K z_L \mid I + J = K + L\}\big) \subset \mathbb{P}^{\binom{n+d}{d}-1}$$

  where $I, J, K, L \in \mathbb{N}^{n+1}$.
- *Let $X \subset \mathbb{P}^n$ be a projective variety. Then $v_d(X) \subset v_d(\mathbb{P}^n) \subset \mathbb{P}^{\binom{n+d}{d}-1}$ is a projective subvariety, and $v_d(X) \cong X$.*
  The main trick in this proof is to note that $\mathbb{V}(F) = \mathbb{V}(x_0F, x_1F, \ldots, x_nF) \subset \mathbb{P}^n$, since not all x_i can vanish simultaneously. So $X = \mathbb{V}(F_1, \ldots, F_r) = \mathbb{V}(G_1, \ldots, G_s)$, where the G_i are all homogeneous of the *same* degree $c \cdot d$ for some c. Thus $G_i = H_i \circ v_d$ for some H_i homogeneous of degree c. Then $v_d(X) = v_d(\mathbb{P}^n) \cap \mathbb{V}(H_1, \ldots, H_s)$.

19.2 Segre embedding

- The **Segre embedding** is the map

$$\sigma_{m,n} \colon (\mathbb{P}^m \times \mathbb{P}^n) \hookrightarrow \mathbb{P}^{mn+m+n}$$
$$([x_0 : \ldots : x_m], [y_0 : \ldots : y_n]) \mapsto [x_0y_0 : x_1y_0 : \ldots : x_my_0 : x_0y_1 : \ldots : x_my_1 : \ldots : x_my_n].$$

  or, equivalently,

$$\sigma_{m,n} \colon \mathbb{P}(k^{m+1}) \times \mathbb{P}(k^{n+1}) \hookrightarrow \mathbb{P}(k^{m+1} \otimes k^{n+1})$$
$$([x, y]) \mapsto [x \otimes y].$$

- *We can think of $k^{m+1} \otimes k^{n+1}$ as matrices, and then $\sigma_{m,n}([x],[y])$ is the matrix product of the column vector x and the row vector y.*
- *The inverse of $\sigma_{m,n}$ is $\pi_c \times \pi_r$, where π_c (respectively π_r) is projection onto any non-zero column (respectively row).*
- The **Segre variety** is $\Sigma_{m,n} = \sigma_{m,n}(\mathbb{P}^m \times \mathbb{P}^n) \subset \mathbb{P}^{mn+m+n}$.
- *The Segre variety is given by*

$$\Sigma_{m,n} = \mathbb{V}\big(\{z_{ij}z_{k\ell} - z_{kj}z_{i\ell} \mid 0 \leqslant i < k < n, 0 \leqslant j < \ell < m\}\big)$$

  i.e. the vanishing of all 2×2 minors of $(z_{ij}) \in \mathrm{Mat}_{(m+1)\times(n+1)}$.

- The **Zariski topology on** $\mathbb{P}^m \times \mathbb{P}^n$ is the subspace topology on $\Sigma_{m,n} \subset \mathbb{P}^{mn+m+n}$, i.e. we declare $\sigma_{m,n}$ and π_r, π_c to be isomorphisms.
- *The Veronese map is given by*

$$\mathbb{P}(k^{n+1}) \hookrightarrow \underbrace{\mathbb{P}(k^{n+1}) \times \ldots \mathbb{P}(k^{n+1})}_{d \text{ times}} \to \mathbb{P}((k^{n+1})^{\otimes d}) \to \mathbb{P}(\mathrm{Sym}^d(k^{n+1}))$$

  where the first map is the diagonal embedding $x \mapsto x \otimes \ldots \otimes x$; the second is given by repeatedly applying the Segre map; and the third is the quotient map.
- *Let $X \subset \mathbb{P}^m$ and $Y \subset \mathbb{P}^n$ be projective varieties. Then $\sigma_{m \times n}(X \times Y) \subset \mathbb{P}^{mn+m+n}$ is a projective variety.*
  This follows from the fact that, if $X = \mathbb{V}(F_1, \ldots, F_r)$ and $Y = \mathbb{V}(G_1, \ldots, G_s)$, then

$$\sigma_{m,n}(X \times Y) = \Sigma_{m,n} \cap \mathbb{V}\big(\{F_k(z_{0j}, \ldots, z_{mj}), G_\ell(z_{i0}, \ldots, z_{in}) \,\big|\, \text{all } i, j, k, \ell\}\big).$$

19.3 Grassmannian, flags, and the Plücker embedding

- The **Grassmannian of d-planes in** k^n $\mathrm{Gr}(d, n)$ is the set of all d-dimensional vector subspaces of k^n.
- $\mathbb{P}^n = \mathrm{Gr}(1, n)$.
- *We can identify $\mathrm{Gr}(d, n)$ with the set of all $(d \times n)$ rank-d matrices modulo $\mathrm{GL}_k(d)$ by associating $V \in \mathrm{Gr}(d, n)$ with the matrix whose rows are any choice of basis for V.*
- The **Flag variety** $\mathrm{Flag}(d_1, \ldots, d_s, n)$ for $0 \leqslant d_1 < d_2 < \ldots < d_s \leqslant n$ is the set of all s-tuples $(V_1, \ldots, V_s)$ where $V_i \subset k^n$ are such that $\dim V_i = d_i$ and $V_i \subset V_{i+1}$. Such tuples are called **flags**.
- The **dth exterior product** $\Lambda^d V$ of a k-vector space V is the vector space of dimension $\binom{\dim V}{d}$ generated by the symbols $v_{i_1} \wedge \ldots \wedge v_{i_d}$ where $i_1 < \ldots < i_d$, and $\{v_{i_j}\}$ is a basis for V. We extend the wedge symbol to all vectors by declaring it to be alternating ($v_i \wedge v_j = -v_j \wedge v_i$) and multilinear.
- The **Plücker map** is defined by

$$\begin{aligned} \mathrm{Gr}(d, n) &\hookrightarrow \mathbb{P}(\Lambda^d k^n) \cong \mathbb{P}^{\binom{n}{d}-1} \\ V &\mapsto k \cdot (v_1 \wedge \ldots \wedge v_d) \qquad \text{where } \{v_i\} \text{ is a basis for } V. \end{aligned}$$

  or equivalently, using the above identification as $(d \times n)$ rank-d matrices modulo $\mathrm{GL}_k(d)$,

$$\begin{aligned} \mathrm{Gr}(d, n) &\hookrightarrow \mathbb{P}^{\binom{n}{d}-1} \\ A &\mapsto [\text{all } d \times d \text{ minors of } A]. \end{aligned}$$

AG 20 Coordinate rings and the Nullstellensatz

20.1 Affine Nullstellensatz

- The **radical** $\sqrt{I}$ of an ideal $I \lhd R$ is defined as $\{r \in R \mid r^k \in I \text{ for some } k \geqslant 1\}$; an ideal I is called **radical** if $\sqrt{I} = I$.
- **Affine Nullstellensatz:** *Let k be an algebraically closed field. Then*
  1. *maximal ideals of $k[x_1, \ldots, x_n]$ are of the form $(x_1 - a_1, \ldots, x_n - a_n) = \mathbb{I}(a)$, where $a = (a_1, \ldots, a_n) \in k^n$;*
  2. *if $J \subset k[x_1, \ldots, x_n]$ is a proper ideal then $\mathbb{V}(J) \neq \varnothing$;*
  3. $\mathbb{I}\mathbb{V}(I) = \sqrt{I}$.
- *Let X be an affine variety. Then $\mathbb{I}(X)$ is a prime ideal iff X is irreducible.*

- *$\mathbb{V}$ and $\mathbb{I}$ form an inclusion reversing bijection*

$$\begin{array}{rcl} \text{radical ideals in } k[x_1, \dots, x_n] & \leftrightarrows & \text{affine varieties in } \mathbb{A}_k^n \\ \text{prime ideals} & \leftrightarrows & \text{irreducible varieties} \\ \text{maximal ideals} & \leftrightarrows & \text{points} \end{array}$$

where, by inclusion reversing, we mean that

$$\begin{aligned} I \subseteq J &\implies \mathbb{V}(J) \subseteq \mathbb{V}(I) \\ X \subseteq Y &\implies \mathbb{I}(Y) \subseteq \mathbb{I}(X). \end{aligned}$$

Work through the proof of this.

20.2 Coordinate rings

- *Ideals $J \triangleleft R/I$ are in bijective correspondence with ideals $\tilde{I} \triangleleft R$ such that $J \subseteq \tilde{I}$.*
- The **coordinate ring** $A(X)$ of an affine variety $X \subseteq \mathbb{A}^n$ is defined as

$$A(X) = k[x_1, \dots, x_n]\Big|_X \cong \frac{k[x_1, \dots, x_n]}{\mathbb{I}(X)}.$$

- Given a finitely-generated reduced k-algebra R we define the **associated variety** X_R as follows: let $a_1, \dots, a_n$ generate R and look at the surjective map $\varphi\colon k[x_1, \dots, x_n] \twoheadrightarrow R$ given by $x_i \mapsto a_i$. Then $k[x_1, \dots, x_n]/\ker\varphi \cong R$, and R is reduced, so $\ker\varphi$ must be radical, and we take $X_R = \mathbb{V}(\ker\varphi) \subseteq \mathbb{A}^n$.
- A commutative ring with identity is said to be **reduced** if it has no **nilpotent elements**, i.e. elements $r \in R \setminus \{0\}$ such that $r^k = 0$ for some $k \geqslant 1$.
- *$k[x_1, \dots, x_n]/I$ is reduced iff I is radical.*
- *There is a contravariant equivalence of categories*

  {affine varieties and morphisms} $\leftrightarrow$ {finitely-generated reduced k-algebras and homomorphisms}

  In particular,

  1. *$X \mapsto A(X)$ and $R \mapsto X_R$ are well defined and inverse to each other, i.e.*
     - *$X \cong Y \implies A(X) \cong A(Y)$;*
     - *$R \cong S \implies X_R \cong X_S$;*
     - *$X \cong X_{A(X)}$;*
     - *$R \cong A(X_R)$;*
  2. *a morphism $F\colon X \to Y$ of affine varieties induces a k-algebra homomorphism $F^\#\colon A(Y) \to A(X)$, and $(F \circ G)^\# = G^\# \circ F^\#$;*
  3. *a k-algebra homomorphism $f\colon R \to S$ induces a morphism $f_\#\colon X_S \to X_R$ of affine varieties, and $(f \circ g)_\# = g_\# \circ f_\#$;*
  4. $(–)_\#$ and $(–)^\#$ are inverses up to isomorphism, i.e. $(F^\#)_\# \cong F$ and $(f_\#)^\# \cong f$.

  The functors $(–)_\#$ and $(–)^\#$ act on morphisms as follows:

  - Let $F\colon X \to Y$ and $g \in A(Y)$. Define $F^\# g = g \circ F$, which is a polynomial since both F and g are.
  - Let $f\colon R \to S$ and choose representations

  $$R = \frac{k[x_1, \dots, x_m]}{I} \quad \text{and} \quad S = \frac{k[y_1, \dots, y_n]}{J}.$$

    Let $F_i \in k[y_1, \dots, y_n]$ be a polynomial representing $f(x_i) \in S$ and define $f_\#\colon \mathbb{A}^n \to \mathbb{A}^m$ by $a \mapsto (F_1(a), \dots, F_m(a))$. We claim that if $a \in \mathbb{V}(J)$ then $f_\#(a) \in \mathbb{V}(I)$, and that if F_i and F_i' are two representatives of $f(x_i)$ then $F_i(a) = F_i'(a)$.

From these definitions we see that, with R, S as above, $a \in \mathbb{A}^n$, and $g \in A(X_R) = R$,

$$\begin{aligned}(f_\#)^\#(g)(a) &= (g \circ f_\#)(a)\\ &= g\big(F_1(a), \ldots, F_m(a)\big)\\ &= f(g)(a)\end{aligned}$$

and similarly for $(F^\#)_\#$.

20.3 Projective Nullstellensatz

- A **graded ring** R is a commutative ring of the form $R = \bigoplus_{d \geqslant 0} R_d$, where each R_d is a subgroup under addition, $R_d \cap R_e = \{0\}$ for $d \neq e$, and $R_d R_e \subseteq R_{d+e}$.
- An ideal $I \triangleleft R$ of a graded ring R is **homogeneous** if any of the following equivalent conditions hold:
  - $I = \bigoplus_{d \geqslant 0}(I \cap R_d)$;
  - I can be generated by homogeneous elements.
- *A homogeneous ideal I is prime iff for all* homogeneous $f, g \in I$, *if $fg \in I$ then $f \in I$ or $g \in I$.*
- *Sums, products, intersections, and radicals of homogeneous ideals are homogeneous.*
- *If R is graded and I homogeneous then R/I is graded.*
- *If $X \subseteq \mathbb{P}^n$ is a projective variety then*

  $$\mathbb{I}(X) = \{F \in k[x_0, \ldots, x_n] \mid F \text{ is homogeneous; } F(x) = 0 \text{ for all } x \in X\}$$

  is a homogeneous ideal.
- The **irrelevant ideal** is $(x_0, \ldots, x_n) \triangleleft k[x_0, \ldots, x_n]$.
- **Projective Nullstellensatz:** *Let k be an algebraically closed field and $J \triangleleft k[x_0, \ldots, x_n]$ a homogeneous ideal. Then*
  1. $\mathbb{V}(J) = \varnothing$ *iff* $(x_0, \ldots, x_n) \subset \sqrt{J}$;
  2. *if* $\mathbb{V}(J) \neq \varnothing$ *then* $\mathbb{I}\mathbb{V}(J) = \sqrt{J}$

  This is a corollary of the affine Nullstellensatz, applied to the affine cone $\widehat{X}$.
- $\mathbb{V}$ *and* $\mathbb{I}$ *form an inclusion reversing bijection*

proper homogeneous radical ideals in $k[x_1, \ldots, x_n]$	$\leftrightarrows$	projective varieties in $\mathbb{P}^n$
homogeneous prime ideals	$\leftrightarrows$	irreducible varieties
irrelevant ideal $(x_0, \ldots, x_n)$	$\leftrightarrows$	the empty set

  where, by inclusion reversing, we mean that

  $$\begin{aligned} I \subseteq J &\implies \mathbb{V}(J) \subseteq \mathbb{V}(I)\\ X \subseteq Y &\implies \mathbb{I}(Y) \subseteq \mathbb{I}(X).\end{aligned}$$

- *Let $X \subseteq \mathbb{A}^n$ be an affine variety, and define $\tilde{I} \triangleleft k[x_1, \ldots, x_n]$ to be the ideal generated by the homogenisation of* all of the *elements of $\mathbb{I}(X)$. Then $\overline{X} = \mathbb{V}(\tilde{I}) \subseteq \mathbb{P}^n$ is the projective closure of $X = \overline{X} \cap U_0$.*

  Work through the proof of this (see p. 26 of lecture notes).

20.4 Homogeneous coordinate rings

- The **homogeneous coordinate ring** $S(X)$ **of a projective variety** $X \subset \mathbb{P}^n$ is the coordinate ring of its cone: $S(X) = k[x_0, \ldots, x_n]/\mathbb{I}(X) = A(\widehat{X})$.
- *The projective Nullstellensatz defines a bijection*

  $$\{\text{projective varieties with an embedding into } \mathbb{P}^n\}$$
  $$\leftrightarrow$$
  $$\{\text{reduced f.g. } k\text{-algebras generated by } n+1 \text{ degree-1 elements of with a representation}\}$$

20.5 Maximal spectrum

- Let R be a finitely-generated reduced k-algebra, i.e. $R = A(X)$ for some affine variety $X \subseteq k[x_1, \dots, x_n]$. Define the **maximal spectrum** $\mathrm{mSpec}\, R = \{m \triangleleft R \mid m \text{ maximal}\}$.
- $\mathrm{mSpec}\, R = X = X_R$ *as sets under the identifications* $a \mapsto \mathfrak{m} = \{f \in R \mid f(a) = 0\}$ *and* $m \mapsto \mathbb{V}(m)$.
- The **Zariski topology on** $\mathrm{mSpec}\, R$ is the topology whose closed sets are the $\mathbb{V}(I) = \{m \in \mathrm{mSpec}\, R \mid I \subseteq m\}$ for $I \triangleleft R$, where we think maximal ideals as points in $X = \mathbb{V}\mathbb{I}(X)$ by using the affine Nullstellensatz.
- $\mathrm{mSpec}\, R \cong X_R$ *as topological spaces.*

AG 21 Categorical quotients

21.1 Definitions and theorems

- An **affine algebraic group** is an affine variety G with a group structure on its points, such that multiplication $\mu \colon G \times G \to G$ and inversion $(-)^{-1} \colon G \to G$ are morphisms of affine varieties.
- An **action of G on X**, where G is an affine algebraic group and X an affine variety, is a morphism of affine varieties $G \times X \to X$ with the usual properties for a group acting on a set.
- Let G be an affine algebraic group acting on an affine variety X, and let Y an arbitrary affine variety. Then a map $F \colon X \to Y$ is a **categorical quotient** if
  1. F is a morphism of affine varieties;
  2. F is constant on the orbits of the group action;
  3. if $F' \colon X \to Y'$ is another morphism constant on the orbits then F' factors uniquely through F.
- Given an action of G on X the **induced G-action $(-)^g$ on** $A(X)$ is given by $f^g(x) = f(gx)$.
- An affine algebraic group is said to be **reductive** if every representation of G is reducible, i.e. is the direct sum of irreducible representations.
- *Let G be a reductive algebraic group and X an affine variety that admits a G-action. If $F \colon X \to Y$ is a morphism of affine varieties that is constant on the G-orbits then $F^\# \colon A(Y) \to A(X)^G$ is surjective, i.e. the image is the G-invariant subring of $A(X)$.*
- A G-action on an affine variety X is said to be **linear** if $g(x + y) = g(x) + g(y)$ for all $x, y \in X$.
- *If G is reductive and acts linearly on X then*
  1. $A(X)^G$ *is a finitely-generated reduced k-algebra;*
  2. *the map* $\alpha_\# \colon X \to \mathrm{mSpec}\left(A(X)^G\right)$ *associated to the embedding* $\alpha \colon A(X)^G \hookrightarrow A(X)$ *is a categorical quotient of affine varieties.*

  The first statement comes without proof. To show the second statement, we first show that $\alpha_\#$ is constant on orbits. Assume not, so that $\alpha_\#(x) \neq \alpha_\#(gx)$ for some g, x.

  Lemma: if $a, b \in Y$ are points of an affine variety and $f(a) = f(b)$ for all $f \in A(Y)$ then $a = b$. Proof: If $a \neq b$ then $a_i \neq b_i$ for some i and so $x_i(a) \neq x_i(b)$.

  So there exists an $f \in A(\mathrm{mSpec}\,(A(X)^G)) = A(X)^G$ such that $f(\alpha_\#(x)) \neq f(\alpha_\#(gx))$. But then
  $$\alpha(f)(x) = (\alpha_\#^\# f)(x) = f(\alpha_\#(x)) \neq f(\alpha_\#(gx)) = (\alpha_\#^\# f)(gx) = \alpha(f)(gx)$$
  which contradicts the fact that $\alpha(f) = f$ (since α is an embedding).

  For the universality, assume that $h \colon X \to Y'$ is constant on orbits. We want to find $\tilde{h} \colon \mathrm{mSpec}\,(A(X)^G) \to Y'$ such that $\tilde{h} \circ \alpha_\# = h$. If $f \in A(Y')$ then $h^\# f(x) = f(h(x)) = f(h(gx)) = h^\# f(gx)$ for all g, x since h is G-invariant. Thus $h^\# f \in A(X)^G$, and so $\alpha \circ h^\# \colon A(Y') \to A(X)$, which induces a morphism that agrees with $h \colon X \to Y'$.

21.2 Examples

1. Some affine algebraic groups include
   - finite groups (discrete points on a variety)
   - $\mathrm{SL}_n(k) = \mathbb{V}(\det -1) \subset \mathbb{A}^{n^2}$
   - $k^\times = k \setminus \{0\}$ thought of as $\mathbb{V}(xy-1) \subset \mathbb{A}^2$
   - $k \cong \mathbb{A}^1$ with an additive structure
2. Some reductive algebraic groups include $k^\times$ and $\mathrm{SL}_n(k)$; the additive group $\mathbb{C}$ is *not* reductive.
3. $k^\times$ acting on $\mathbb{A}^2$ by $t(a,b) = (a/t, bt)$. The induced $k^\times$-action on $k[x,y] = A(\mathbb{A}^2)$ is $t\cdot x = tx$ and $t\cdot y = t^{-1}y$. Then $k[x,y]^{k^\times} = k[xy] \cong k[z]$ with the map $k[z] \hookrightarrow k[x,y]$ being given by $z \mapsto xy$. This defines the quotient map $\mathbb{A}^2 \to \mathbb{A}^1$ by $(a,b) \mapsto ab$, and the orbits are
   - $\{xy = s \neq 0\} \mapsto s$
   - $\{(a,0) \mid a \neq 0\} \mapsto 0$
   - $\{(0,b) \mid b \neq 0\} \mapsto 0$
   - $\{(0,0)\} \mapsto 0$

   Thus the quotient map is not injective on orbits, i.e. the categorical quotient is *not* simply the set of orbits. In fact, two orbits map to the same point iff the closure of the orbits has a non-trivial intersection.

AG

22 Primary decomposition of ideals

22.1 Definitions and theorems

Let R be a commutative ring (with unity).

- Given $x \in R$ we define the **annihilator of** x as $\mathrm{Ann}(x) = \{r \in R \mid rx = 0\}$
- A *proper* ideal $Q \triangleleft R$ is called **primary** if, for all $f, g \in R$ if $fg \in Q$ then $f \in Q$ or $g^n \in Q$ for some $n \geqslant 0$, or equivalently if all zero divisors of R/Q are nilpotent.
- *If Q is a primary ideal then $\sqrt{Q}$ is prime (and in fact is the smallest prime ideal containing Q).*
- Given a prime ideal $P \triangleleft R$, a *primary* ideal $Q \triangleleft R$ is called P**-primary** if $P = \sqrt{Q}$.
- *If Q_1 and Q_2 are both P-primary then $Q_1 \cap Q_2$ is P-primary.*
- A **primary decomposition of an ideal** $I \triangleleft R$ is an equality $I = Q_1 \cap \ldots \cap Q_k$ where each Q_i is primary.
- Given a primary decomposition $I = \bigcap_{i\in J} Q_i$ we say that the Q_i are **irredundant** if no smaller subcollection $\{Q_i\}_{i \in J'}$ for $J' \subsetneq J$ is such that $I = \bigcap_{i \in J'} Q_i$.
- A primary decomposition is called **reduced** if the $P_i = \sqrt{Q_i}$ are all distinct *and* the Q_i are irredundant.
- Given a *reduced* primary decomposition $\{Q_i\}_{i\in J}$ we call the prime ideals $P_i = \sqrt{Q_i}$ the **associated primes of** I.
- The **minimal** (or **isolated**) **prime ideals of** I are the minimal associated primes of I with respect to inclusion; the other associated primes of I are called the **embedded prime ideals of** I.
- The $\mathbb{V}(P_i) \subseteq \mathbb{V}(I)$ for P_i the associated primes of I are called the **associated reduced components of** $\mathbb{V}(I)$, and such a component is called **embedded** if $\mathbb{V}(P_i) \neq \mathbb{V}(I)$.
- *The minimal prime ideals of I are minimal amongst all prime ideals containing I, and all such minimal prime ideals are found as associated primes of I.*

- **Lasker-Noether theorem:** *If R is a Noetherian ring then every ideal has a primary decomposition, and this decomposition can be reduced.*

  Say that an ideal $I \triangleleft R$ is **indecomposable** if $[I = J \cap K \implies I = J$ or $I = K]$ for any ideals J, K. Note that all prime ideals are indecomposable.

  Claim 1: Every ideal $I \triangleleft R$ is the intersection of some indecomposable ideals.

  Proof 1: Let $\mathcal{S}$ denote the set of ideals *not* expressible as the intersection of indecomposable ideals, and assume (for contradiction) that $\mathcal{S} \neq \varnothing$. By Noetherian-ness, there is a maximal element $I \in \mathcal{S}$, and so there exist *strictly larger* ideals J, K such that $I = J \cap K$. But then $J, K \notin \mathcal{S}$, and so we can write them as an intersection of indecomposable ideals, and thus so too with I.

  Claim 2: Every indecomposable ideal is primary.

  Proof 2: Note that $I \triangleleft R$ is indecomposable iff $0 \triangleleft R/I$ is indecomposable (and similarly for primary ideals), so it is enough to prove that if $0 \triangleleft R$ is indecomposable then it is primary. So let $x, y \in R$ with $xy = 0$. Then $y \in \mathrm{Ann}(x)$, and we have the chain $\mathrm{Ann}(x) \subset \mathrm{Ann}(x^2) \subset \ldots$, which eventually stabilises at $\mathrm{Ann}(x^n)$ for some n by Noetherian-ness. But $(x^n) \cap (y) = 0$, so if 0 is indecomposable then $x^n = 0$ or $y = 0$. (To see that $(x^n) \cap (y) = 0$ let $a \in (x^n) \cap (y)$ so that $a = cy = dx^n$, so that $ax = cxy = 0$ and $0 = ax = dx^{n+1}$, so $d \in \mathrm{Ann}(x^{n+1}) = \mathrm{Ann}(x^n)$, so $a = dx^n = 0$.)

  As for the reduction statement, note that if $\sqrt{Q_i} = \sqrt{Q_j} = P_i$ then we can replace them both by $Q_i \cap Q_j$, since this is also P_i-primary.

- **Uniqueness theorem:** *The associated prime ideals of I are uniquely determined.*

  Note that $P_1, \ldots, P_k$ are exactly the prime ideals of R which are the annihilator of some point in R/I, i.e. $P_i = \mathrm{Ann}(x_i)$, and are thus uniquely determined by I.

22.2 Examples

Work through some examples (see the lecture notes for some).

AG

23 Discrete invariants

23.1 Dimension

23.1.1 Geometric dimension

Let X be a variety (affine or projective).

- A **chain of length** m is a *strict* chain of inclusions $X_0 \subsetneq X_1 \subsetneq \ldots \subsetneq X_m \subseteq X$ where each X_i is *irreducible*. Note that we can always start with $X_0 = \{p\}$ for some point $p \in X$, and if X is irreducible then we can end with $X_m = X$.
- The **local dimension** $\dim_p X$ **of** X **at a point** $p \in X$ is the maximum over all lengths of chains starting with $X_0 = \{p\}$.
- *The local dimension of X at p is equal to the dimension of the irreducible component of X containing p.*
- The **dimension** $\dim X$ **of** X is the maximum over all lengths of chains, or equivalently the maximum over the local dimension at each point, i.e. $\dim X = \max_{p \in X} \dim_p X$.
- We say that X has **pure dimension** if $\dim_p X = \dim_q X$ for all points $p, q \in X$.
- The **codimension** $\mathrm{codim}_X Y$ **of an *irreducible* subvariety** $Y \subset X$ is the maximum over all lengths of chains starting with $X_0 = Y$ and *not ending with* X.

1. *An affine variety $X = \mathbb{V}(I) \subset \mathbb{A}^n$ is a finite set of points iff $A(X)$ is a finite-dimensional k-vector space; and if so then $|X| = \dim_k A(X)$, i.e. the dimension of $A(X)$ as a k-vector space.*

   The proof of this uses the primary decomposition theorem and the Chinese remainder theorem

2. *If $X \subseteq Y$ are both varieties (affine or projective) then $\dim X \leqslant \dim Y$. Further, if both X and Y are irreducible and $X \subsetneq Y$ then $\dim X = \dim Y$ (and, in particular, if $\dim X = \dim Y$ then $X = Y$).*

23.1.2 Algebraic dimension

Let R be a commutative ring (with identity).

- A **chain of length** m is a *strict* chain of inclusions $\mathfrak{p}_m \subsetneq \mathfrak{p}_{m-1} \subsetneq \ldots \subsetneq \mathfrak{p}_1 \subsetneq \mathfrak{p}_0 \subset R$ where each $\mathfrak{p}_i \triangleleft R$ is a prime ideal. Note that we can always start with a maximal ideal $\mathfrak{m}$, and if R is an integral domain then we can always end with $\mathfrak{p}_m = \{0\}$.
- *If R is Noetherian then the descending chain condition holds for* prime *ideals.*
- The **height** $\mathrm{ht}(\mathfrak{p})$ **of a prime ideal** is the maximal length of a chain with $\mathfrak{p}_0 = \mathfrak{p}$.
- The **Krull dimension** $\dim R$ **of a ring** R is the maximal height over all *maximal* ideals.
- The **height** $\mathrm{ht}(I)$ **of an arbitrary ideal** $I \triangleleft R$ is the minimum over the heights of all prime ideals $\mathfrak{p}$ containing I.

1. $\mathrm{ht}(\mathfrak{p}) = \dim R_{\mathfrak{p}}$
2. $\mathrm{ht}(I) = \mathrm{codim}_{\mathrm{Spec}\, R}\, \mathbb{V}(I)$.
3. *The minimal prime ideals are exactly those of height zero.*
4. **Krull's principal ideal theorem:** *If R is Noetherian and $f \in R$ is neither a unit nor a zero-divisor then $\mathrm{ht}((f)) = 1$.*
5. **Krull's height theorem:** *If R is Noetherian and $(f_1, \ldots, f_m) \neq R$ then $\mathrm{ht}((f_1, \ldots, f_m)) \leqslant m$. In particular, $\mathrm{ht}(\mathfrak{p})$ is at most the number of generators of $\mathfrak{p}$, and conversely if $\mathfrak{p}$ is a prime ideal of height m then $\mathfrak{p}$ is a minimal prime ideal over some ideal generated by m elements.*

23.1.3 Equivalence of geometric and algebraic dimension

1. *If $X \subset \mathbb{A}^n$ is an affine variety then $\dim X = \dim A(X)$.*
2. *If $X \subset \mathbb{P}^n$ is a projective variety then $\dim X$ is equal the maximal length of chains of* homogeneous *prime ideals which do* not *contain the irrelevant ideal.*
3. *For a maximal length chain $\{\mathfrak{p}_i\}$ of (homogeneous) prime ideals,*

$$\mathrm{ht}(\mathfrak{p}_i) = \mathrm{codim}\, \mathbb{V}(\mathfrak{p}_i) = n - \dim \mathbb{V}(\mathfrak{p}_i)$$

   where n is the dimension of the ambient space (either $\mathbb{A}^n$ or $\mathbb{P}^n$).
4. *If $X \subset \mathbb{A}^n$ is an irreducible affine variety then $\dim X = n - 1$ iff $X = \mathbb{V}(f)$ for some irreducible $f \in k[x_1, \ldots, x_n]$. (The corresponding statement for projective varieties holds, with f homogeneous.)*

   The fiddly direction is assuming that $\dim X = n - 1$: this implies that $\mathbb{I}(X) \neq (0)$ and so there exists some non-zero $f \in \mathbb{I}(X)$, which we can assume to be irreducible, and thus prime (R is a UFD), since $\mathbb{I}(X)$ is prime (X is irreducible). Then $X \subseteq \mathbb{V}(f) \subsetneq \mathbb{A}^n$, so by a dimension argument we see that $X = \mathbb{V}(f)$.

23.2 Degree

23.2.1 Definitions and theorems

Let $X \subset \mathbb{P}^n$ be a projective variety.

- A **linear subvariety of** $\mathbb{P}^n$ is a projectivisation $L = \mathbb{P}(\widehat{L})$ for some vector subspace $\widehat{L} \subset \mathbb{A}^{n+1}$.
- The **degree** $\deg(X)$ of a projective variety X is the maximum number of intersections of X with L over all linear subvarieties $L \subset \mathbb{P}^n$ with $\dim L + \dim X = n$, i.e. $\widehat{L} \in \mathrm{Gr}(n + 1 - \dim X, n + 1)$.

- *This maximum is attained for 'almost all' L, i.e. the L where one gets fewer intersections (or infinitely many) form a proper closed subset of the Grassmannian.*
- *The degree of a projective variety depends on its embedding.*
- The **degree** $\deg(Y)$ of an *affine* variety is defined to be the degree $\deg(\overline{Y})$ of the projective closure of $Y \subset U_0 \subset \mathbb{P}^n$.
- *Let $F \in k[x_0, \ldots, x_n]$ be homogeneous of degree d with no repeated factors. Then $\deg \mathbb{V}(F) = d$.*
  Let L be any line in $\mathbb{P}^n$ and $X = \mathbb{V}(F)$. Then $X \cap L = \mathbb{V}(F|_L) \subset L \cong \mathbb{P}^1$. After a linear change of coordinates we can assume that $L = \mathbb{V}(x_2, \ldots, x_n)$, and so $F|_L$ is generally a degree d polynomial in x_0, x_1, and so generally has d zeros.
- **Weak Bézout's Theorem:** *Let $X, Y \subset \mathbb{P}^n$ be projective varieties of* pure dimension *with $\dim(X \cap Y) = \dim X + \dim Y - n$. Then $\deg(X \cap Y) \leqslant (\deg X)(\deg Y)$.*

23.2.2 Examples

1. A hyperplane H has $\deg H = 1$, e.g. $\mathbb{V}(x_0) \cap \mathbb{V}(x_2, \ldots, x_n) = \{[0 : 1 : 0 : \ldots : 0]\}$.
2. $\mathbb{P}^1 \subset \mathbb{P}^1$ has $\deg \mathbb{P}^1 = 1$, by taking L to be any point.
3. Say $X = \mathbb{V}(xz - y^2) \subset \mathbb{P}^2$. Then the linear subvarieties $L = \mathbb{V}(ax + by + cz)$ are in bijection with planes $\widehat{L} \subset \mathbb{A}^3 \in \mathrm{Gr}(2, 3)$, and these are in bijection with lines normal to the plane, which are of the form $[a : b : c] \in \mathbb{P}^2$. Suppose that $c \neq 0$, then $x \neq 0$ (otherwise $y = 0$, so $z = 0$, and there are no intersections), so scale to $x = 1$. If $b \neq 0$ also then $y = \frac{-cz-a}{b}$ and $z = y^2$ gives two intersections if the discriminant $b^2(b^2 - 4ac)$ is non-zero. So $\deg X = 2$, and the set of 'bad' $L = [a : b : c] \in \mathbb{P}^2$ forms a subset of $\mathbb{V}(c) \cup \mathbb{V}(b) \cup \mathbb{V}(b^2(b^2 - 4ac)) \subsetneq \mathbb{V}(bc(b^2 - 4ac))$.
4. The Veronese embedding $\mathbb{V}(xz - y^2) \cong \mathbb{P}^1$ is such that $\deg \mathbb{V}(xz - y^2) = 2$ but $\deg \mathbb{P}^1 = 1$. This shows that the degree is dependent on the embedding.

23.3 Hilbert function

Let $X \subset \mathbb{P}^n$ be a projective variety.

- Recall that $S(X) = A(\widehat{X}) = \bigoplus_{m \geqslant 0} S(X)_m$, where $S(X)_m = k[x_0, \ldots, x_n]/\mathbb{I}(X)_m$ is the degree-m part.
- Define the **Hilbert function** h_X **of** X by

$$h_X(m) = \dim_k S(X)_m = \binom{m+n}{m} - \dim \mathbb{I}(X)_m$$

  for $m \in \mathbb{N}$.
- *There exists $p_X \in k[x]$ and $m_0 \in \mathbb{N}$ such that for all $m \geqslant m_0$ we have $h_X(m) = p_X(m)$. This polynomial p_X is called the* **Hilbert polynomial of** X.
- *The Hilbert polynomial depends on the embedding of X.*
- *The leading term of the Hilbert polynomial is $\left(\frac{\deg X}{\dim X!}\right) \cdot m^{\dim X}$.*
- *If $X, Y \subset \mathbb{P}^n$ are linearly equivalent then $p_X = p_Y$.*
- A **flat family of varieties** is a projective variety $X \subset \mathbb{P}^n$ together with a surjective morphism $\pi \colon X \twoheadrightarrow B$ where B is an irreducible projective (or quasi-projective) variety and the fibres $X_b = \pi^{-1}(b)$ all have the same Hilbert polynomial.

AG 24 Quasi-projective varieties and regular maps

24.1 Quasi-projective varieties

- A **quasi-projective variety** $X \subset \mathbb{P}^n$ is any *open* subset of a projective variety, i.e. X is **locally closed**, i.e. X is the intersection of an open and a closed subset of $\mathbb{P}^n$, i.e.

$$X = (\mathbb{P}^n \setminus \mathbb{V}(J)) \cap \mathbb{V}(I).$$

Note that this includes affine varieties ($X = \mathbb{A}^n \cap \overline{X}$) and projective varieties ($X = \mathbb{P}^n \cap X$).

- A **morphism of quasi-projective varieties** $F \colon X \to Y$ is a map given locally by $F(p) = [F_0(p) : \ldots : F_n(p)]$ for homogeneous polynomials F_i of the *same* degree.
- *Let $X, Y \subset \mathbb{A}^n$ be quasi-projective varieties. If there are mutually inverse polynomial maps $X \leftrightarrows Y$ then $X \cong Y$ as quasi-projective varieties. (The converse is not necessarily true.)*
- A quasi-projective variety X is **affine** if it is isomorphic (as a quasi-projective variety) to an affine variety.
- *Let $X \subset \mathbb{A}^n$ be an affine variety and $f \in A(X)$. Then $D_f = X \setminus \mathbb{V}(f)$ is an affine quasi-projective variety, with*

$$A(D_f) = A(X)_f = A(X)\left[\frac{1}{f}\right]$$

i.e. the localisation at f.

Define $\widetilde{I} = (\mathbb{I}(X), x_{n+1}f - 1) \lhd k[x_1, \ldots, x_n, x_{n+1}]$. Then $\mathbb{V}(\widetilde{I}) \subset \mathbb{A}^{n+1}$ is affine with $A(\mathbb{V}(\widetilde{I})) = A(X)[x_{n+1}]/(x_{n+1}f - 1) = A(X)_f$. We claim that $\varphi \colon D_f \to \mathbb{V}(\widetilde{I})$ is an isomorphism of quasi-projective varieties, where $\varphi \colon a = (a_1, \ldots, a_n) \mapsto \left(a_1, \ldots, a_n, \frac{1}{f(a)}\right)$ with $\varphi^{-1} \colon (b_1, \ldots, b_n, b_{n+1}) \mapsto (b_1, \ldots, b_n)$. Embed $D_f \hookrightarrow \mathbb{P}^n$ via $(a_1, \ldots, a_n) \mapsto [1 : a_1 : \ldots : a_n]$ and $V(\widetilde{I}) \hookrightarrow \mathbb{P}^{n+1}$ via $(b_1, \ldots, b_{n+1}) \mapsto [1 : b_1 : \ldots : b_{n+1}]$. Then φ is the restriction of $F \colon \mathbb{P}^n \to \mathbb{P}^{n+1}$ given by $F \colon [a_0 : \ldots : a_n] \mapsto [a_0\tilde{f}(a) : \ldots : a_{n-1}\tilde{f}(a) : a_0^{1+\deg f}]$, where $\tilde{(}f)(a) = a_0^{\deg f} f\left(\frac{a_1}{a_0}, \ldots, \frac{a_n}{a_0}\right)$.

- *Every quasi-projective variety has a finite open cover by affine quasi-projective subvarieties. In particular, the affine open subsets form a basis for the topology.*

Say that $X \subset \mathbb{P}^n$ is quasi-projective, and so of the form $X = \mathbb{V}(F_1, \ldots, F_r) \setminus \mathbb{V}(G_1, \ldots, G_s)$. We only need to check the claim on the open $U_i \cap X$. But $U_0 \cap X = \mathbb{V}(f_1, \ldots, f_r) \setminus \mathbb{V}(g_1, \ldots, g_s)$, where $f_i(a) = F_i(1, a)$ and similarly for g_i, i.e. $f_i = F_i|_{x_0=1}$. Then $U_0 \cap X = \bigcup_j \mathbb{V}(f_1, \ldots, f_r) \setminus \mathbb{V}(g_j) = \bigcup_j (D_{g_j} \cap \mathbb{V}(f_1, \ldots, f_r))$.

24.2 Regular functions

Let X be an affine variety and $U \subset X$ an open subset.

- A function $f \colon U \to k$ is **regular at a point** $p \in U$ if there exists an open set $W \subset U$ containing p and such that $f = \frac{g}{h}$ on W for some $g, h \in A(X)$ with $h(w) \neq 0$ for any $w \in W$ (or equivalently just $h(p) \neq 0$).
- A function $f \colon U \to k$ is **regular** if it is regular at each $p \in U$. **We write** $\mathcal{O}_X(U)$ to denote the collection of regular functions on U.
- If X is instead a *quasi-projective variety* then a map $F \colon U \to k$ is **regular at a point** $p \in U$ if there exists some *affine open* $W \subset U$ containing p with $F|_W$ regular at p. If F is regular at each $p \in U$ then we say that it is **regular**.
- *Let X be an affine variety. Then $\mathcal{O}_X(X) = A(X)$.*

(We only prove the case when X is irreducible.) Clearly if $f \in A(X)$ then $f = \frac{f}{1}$ on all of X, so it is regular. Let $f \in \mathcal{O}_X(X)$, so that for all $p \in X$ there exists open $U_p \subset X$ containing p with $f = \frac{g_p}{h_p}$ with $h_p \neq 0$ on U_p. We can take $U_p = D_{F_p}$ for some $F_p \in A(X)$ (since these form a basis), and since X is compact (by the Hilbert basis theorem) we have a finite subcover $D_{F_{p_1}} \cup \ldots \cup D_{F_{p_m}}$. (Write F_i to mean F_{p_i}, similarly for f_i, g_i, D_i, etc.) On $D_i \cap D_j$ we have $g_i h_j = g_j h_i$. Since X is irreducible, $D_i \cap D_j \subset X$ is a dense open subset, and so $X = \overline{D_i \cap D_j} \subset \mathbb{V}(g_i h_j - g_j h_i)$. Thus $g_i h_j = g_j h_i$ on all of X. But at each point $p \in X$ there is at least one h_i that is non-zero, and so $\mathbb{V}(h_1, \ldots, h_m) = \varnothing$. So, by the Nullstellensatz, $(h_1, \ldots, h_m) = (1)$, thus $1 = \sum \alpha_i h_i$ for some $\alpha_i \in A(X)$. Then $f|_{D_j} = 1 \cdot \frac{g_j}{h_j} = \sum_i \alpha_i h_i \frac{g_j}{h_j} = \sum_i \alpha_i h_j \frac{g_i}{h_j} = \sum_i \alpha_i g_i \in A(X)$. So $f = \sum_i \alpha_i g_i \in A(X)$ on all of X.

- *Let $X \subset \mathbb{A}^n$ be an affine variety and $D_h \subset X$ for some $h \in A(X)$. Then $\mathcal{O}_X(D_h) = A(X)[\frac{1}{h}] \cong A(X)_h$.*
- Let X, Y be quasi-projective varieties. A map $F\colon X \to Y$ is a **regular map** if for all $p \in X$ there exist open affine subvarieties $U \subset X$ and $V \subset Y$ with $p \in U$ and $F(p) \in V$ such that $F(U) \subset V$ and $F|_U \colon U \to V$ is defined by regular functions.
- *A map $F\colon X \to Y$ between quasi-projective varieties is a regular map iff it is a morphism of quasi-projective varieties.*

  For an affine open $U \subset X$ there is an affine variety $Z \subset \mathbb{A}^n$ such that $U \cong Z$. Then it can be shown that $\mathcal{O}_X(U) \cong \mathcal{O}_Z(Z) = A(Z)$. Therefore a map defined by regular functions is locally a polynomial map.
- For a quasi-projective variety X, the **ring of germs $\mathcal{O}_{X,p}$ of regular functions at** p, or the **stalk of $\mathcal{O}_X$ at** p, is the set of pairs (f, U), where $U \subset X$ is open and contains p, and $f\colon U \to k$ is regular at p, modulo the equivalence relation $(f, U) \sim (f', U')$ if $f|_W = f'|_W$ for some open $W \subset U \cap U'$ containing p.
- A morphism $F\colon X \to Y$ of quasi-projective varieties **induces a ring homomorphism on stalks**, denoted $F_p^*\colon \mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}$ given by $(g, V) \mapsto \big(F^*g, F^{-1}(V)\big)$ where $F^*\colon \mathcal{O}_Y(V) \to \mathcal{O}_X\big(F^{-1}(V)\big)$ is given by $F^*g = g \circ F$.
- *If $F, G\colon X \to Y$ are such that $F_p^* = G_p^*$ for all $p \in X$ then $F = G$.*

AG 25 Function fields and rational maps

25.1 Function fields

- For an *irreducible affine* variety $X \subset \mathbb{A}^n$, since $A(X)$ is an integral domain, we can define the **function field** $k(X)$ as the field of fractions of the coordinate ring, i.e. $\{\frac{g}{h} \mid g, h \in A(X)\}/\sim$, where $\frac{g}{h} \sim \frac{g'}{h'}$ if $gh' = g'h$.
- *Let $U, U' \subset X$ be* non-empty *affine opens in an* irreducible *affine variety X. Then, for all* non-empty *basic opens $D_h \subset U \cap U'$, we have that $k(U) \cong k(D_h) \cong k(U')$.*
- For an *irreducible quasi-projective* variety X and a *non-empty* affine open $U \subset X$ we **define** $k(X) = k(U)$. This is independent of the choice of U, by the above lemma.
- For an *irreducible affine variety X* we have the following relations of rings inside the function field:

$$A(U) = \mathcal{O}_X(U) = \bigcap_{D_h \subset U} \mathcal{O}_X(D_h) = \bigcap_{p \in U} \mathcal{O}_{X,p} \quad \subset \quad \mathcal{O}_{X,p} = A(X)_{\mathfrak{m}_p} \quad \subset \quad A(X)_{(0)} = k(X).$$

25.2 Rational maps

- For an *irreducible quasi-projective* variety X and an arbitrary quasi-projective variety Y, a map $f\colon X \dashrightarrow Y$ is called a **rational map** if it is defined on some non-empty open subset of X and is regular where it is defined. We identity rational maps which agree on some non-empty open subset, and often write them as an equivalence class of a pair: $f = [(F, U)]$, where $F\colon U \to Y$ is a regular map, and so we can always assume that $F\colon U \to V \subset Y$ is *polynomial* between affine opens U, V. Note that rational maps may not necessarily compose.
- A **rational function** is a rational map $f\colon X \dashrightarrow \mathbb{A}^1$.
- Note that if X is an affine variety then $f \in A(X)$ is simply a rational function $f\colon X \dashrightarrow \mathbb{A}^1$.
- *Let X be an irreducible quasi-projective variety. Then $k(X) \cong \{$rational functions on $X\}$*

  Restricting to a non-empty open affine in X we may assume that X is an irreducible affine variety. We can pick an arbitrarily small subset $D_h \subset U \subset X$ of the open subset on which our rational function $f = \frac{g}{h} \in k(D_h)$ is defined. But we've already shown that $\frac{g}{h} \in k(D_h) \cong k(X)$ is uniquely defined, and is independent of the choice of D_h.

- A rational map $f = [(F, U)]\colon X \dashrightarrow Y$ is **dominant** if the image $F(U) \subset Y$ is dense.
- A **birational equivalence** $f\colon X \dashrightarrow Y$ is a dominant rational map between quasi-projective varieties which has a rational inverse, i.e. there exists a rational map $g\colon Y \dashrightarrow X$ such that $f \circ g = \mathrm{id}_Y$ and $g \circ f = \mathrm{id}_X$, where equality means equality on some *non-empty* open subset. We say that such an X, Y are **birational**, and write $X \simeq Y$.
- A quasi-projective variety X is **rational** if it is birational to $\mathbb{A}^n$ for some n.
- *Let X, Y be* irreducible affine *varieties. Then a rational map $f\colon X \dashrightarrow Y$ determines a k-algebra homomorphism $f^*\colon A(Y) \to k(X)$ given by $f^* y = y \circ f$. Further, f^* is injective iff f is dominant, in which case $f^*\colon k(Y) \to k(X)$ and is given by $\frac{g}{h} \mapsto \frac{f^* g}{f^* h}$*
  Let $y \in Y$ be non-zero. Then $[F^* y = 0] \iff [y(F(u)) = 0\ \forall u \in U] \iff [F(u) \in \mathbb{V}(y)\ \forall u \in U] \iff [F(U) \subset \mathbb{V}(y) \subsetneq Y]$. But $F(U)$ is not dense iff $F(U) \subseteq \mathbb{V}(J)$ for some J, so taking non-zero $y \in J$ and applying the above gives the required result. For the final claim, note that $f^* h \neq 0$ if $h \neq 0$, since f^* is injective.

AG 26 Irreducible q.p. varieties and f.g. field extensions

There is an equivalence of categories:

$$\{\text{irred. q.p. varieties } X \text{ with rational dominant maps}\} \leftrightarrow \{\text{f.g. field extensions } k \hookrightarrow K \text{ with } k\text{-alg. homs.}\}^{\mathrm{op}}$$

$$X \mapsto k(X)$$

$$(f = \varphi^*\colon X \dashrightarrow Y) \mapsto (\varphi = f^*\colon k(Y) \to k(X)).$$

In particular, the following hold:

- *$f^{**} = f$ and $\varphi^{**} = \varphi$;*
- *$(g \circ f)^* = f^* \circ g^*\colon k(Z) \to k(Y) \to k(X)$ for $g \circ f\colon X \dashrightarrow Y \dashrightarrow Z$, and conversely for φ;*
- *$X \simeq Y$ iff $k(X) \cong k(Y)$.*

The proof is split into steps:

1. f induces $\varphi = f^*$, i.e. the functor is fully faithful.
2. For field extensions $k \hookrightarrow A$, $k \hookrightarrow B$, any k-algebra homomorphism $A \to B$ is a field extension, i.e. injective.
3. For X, Y irreducible *affine* varieties, a k-algebra homomorphism $\varphi\colon k(Y) \to k(X)$ determines a birational $f\colon X \dashrightarrow Y$.
4. For X, Y *quasi-projective* varieties, a k-algebra homomorphism $\varphi\colon k(Y) \to k(X)$ determines a birational $f\colon X \dashrightarrow Y$.
5. For any finitely-generated $k \hookrightarrow K$ there exists an irreducible quasi-projective variety X with $K \cong k(X)$, i.e. the functor is essential surjective.
6. The functor forms an equivalence of categories.

Fill in the details of the above proof.

As a corollary, any irreducible affine *variety is birational to a hypersurface in some* affine *space.*

AG 27 Localisation theory

27.1 Algebraic localisation

Let R be a commutative ring (with identity).

- A subset $S \subset R$ is **multiplicative** if $1 \in S$ and $S \cdot S \subseteq S$.

- The **localisation** of R as S is $S^{-1}R = (R \times S)/\sim$, where $(r,s) \sim (r',s')$ if there exists $t \in S$ such that $t(rs' - r's) = 0$. We write pairs (r,s) as $\frac{r}{s}$.
- *$S^{-1}R = 0$ iff $0 \in S$.*
- For $f \in R$ we write R_f to mean $S^{-1}R$ where $S = \{1, f, f^2, \dots\}$.
- For $\mathfrak{p} \lhd R$ prime we write $R_\mathfrak{p}$ to mean $S^{-1}R$ where $S = R \setminus \mathfrak{p}$.
- *If R is an integral domain then $R_f = R[\frac{1}{f}]$.*
- *The **canonical ring homomorphism** $\pi\colon R \to S^{-1}R$ given by $r \mapsto \frac{r}{1}$ is injective if R is an integral domain and $0 \notin S$.*
- We say that R is a **local ring** if it has a *unique* maximal ideal $\mathfrak{m} \subset A$.
- *R is local iff there exists a* proper *ideal $I \lhd R$ such that all elements in the complement $R \setminus I$ are units.*
- *If $\mathfrak{p} \lhd R$ is prime then $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p} = \{\frac{r}{s} \mid r \in \mathfrak{p}, s \notin \mathfrak{p}\}$.*
- *If R is an integral domain then $R = \bigcap_\mathfrak{m} R_\mathfrak{m} = \bigcap_\mathfrak{p} R_\mathfrak{p}$, where the intersections are taken over all maximal ideals $\mathfrak{m}$ and prime ideals $\mathfrak{p}$.*
- *There is a bijective correspondence*

  $$\{\mathfrak{p} \lhd R \mid \mathfrak{p} \cap S = \varnothing\} \leftrightarrow \{\hat{\mathfrak{p}} \lhd S^{-1}R\}$$
  $$\mathfrak{p} \mapsto \hat{\mathfrak{p}} = \mathfrak{p} \cdot S^{-1}R = \left\{\frac{p}{s} \mid p \in \mathfrak{p}, s \in S\right\}$$
  $$\mathfrak{p} = \pi^{-1}(\hat{\mathfrak{p}}) = \{r \in R \mid \frac{r}{1} \in \hat{\mathfrak{p}}\} \leftarrow\!\shortmid \hat{\mathfrak{p}}.$$

  In particular, for a fixed prime ideal $\mathfrak{q} \lhd R$, there is the bijective correspondence

  $$\{\mathfrak{p} \subset \mathfrak{q} \lhd R\} \leftrightarrow \{\hat{\mathfrak{p}} \lhd R_\mathfrak{q}\}$$
  $$\mathfrak{p} = \pi^{-1}(\hat{\mathfrak{p}}) \leftrightarrow \hat{\mathfrak{p}} = \mathfrak{p}R_\mathfrak{q}$$

27.2 Geometric localisation

Let X be an *affine* variety.

- *For a point $p \in X$ the stalk $\mathcal{O}_{X,p}$ of the structure sheaf $\mathcal{O}_X$ is $\mathcal{O}_{X,p} \cong A(X)_{\mathfrak{m}_p}$ where $\mathfrak{m}_p = \mathbb{I}(p)$ is the maximal ideal corresponding to p.*

  The isomorphism is defined by $(f, U) \mapsto \frac{g}{h}$ for $h(p) \neq 0$, where $f|_U = \frac{g}{h}$, with inverse map $\frac{g}{h} \mapsto (\frac{g}{h}, D_h)$ for $h \notin \mathfrak{m}_p$.
- *There is a bijective correspondence*

  $$\{\text{subvarieties } Y \subset X \text{ passing through } p\} \leftrightarrow \{\mathfrak{p} \lhd \mathcal{O}_{X,p}\}$$
  $$Y = \mathbb{V}(\mathfrak{p}) \leftrightarrow \mathfrak{p} = \{f \in \mathcal{O}_{X,p} \mid f(Y) = 0\}.$$

  In particular, the point $Y = \{p\}$ corresponds to the unique *maximal ideal $\mathfrak{m}_p\mathcal{O}_{X,p}$.*

  This follows from the bijective correspondence in the previous section on localisation.

27.3 Homogeneous localisation

Let $R = \bigoplus_{m \geqslant 0} R_m$ be an $\mathbb{N}$-graded ring.

- Let $S \subset R$ be a multiplicative set *consisting only of homogeneous elements* Then $S^{-1}R = \bigoplus_{m \in \mathbb{Z}} (S^{-1}R)_m$ has a $\mathbb{Z}$-grading: for homogeneous elements $r \in R, s \in S$ **define** $m = \deg \frac{r}{s} = \deg r - \deg s$.
- The **homogeneous localisation** is the subring $(S^{-1}R)_0$ of $S^{-1}R$.
- We write $R_{(f)}$ **to mean** $(R_f)_0$ (where f is *homogeneous*), and $R_{(\mathfrak{p})}$ **to mean** $(R_\mathfrak{p})_0$.
- *Let $X \subset \mathbb{P}^n$ be a quasi-projective variety and $\overline{X}$ its projective closure. Write $X_0 = \overline{X} \cap U_0$ and let $p \in X_0$. Then*

  $$\mathcal{O}_{X,p} \cong A(X_0)_{\mathfrak{m}_{p,0}} \cong S(\overline{X})_{(\mathfrak{m}_p)}$$

  where $\mathfrak{m}_{p,0} = \{f \in A(X_0) \mid f(p) = 0\}$ and $\mathfrak{m}_p = \{F \in S(\overline{X}) \mid F(p) = 0\}$.

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28 Tangent spaces and smooth points

28.1 Tangent spaces

Let X be an *affine* variety.

- Let $F \in k[x_1, \dots, x_n]$ and $p = (p_1, \dots, p_n) \in \mathbb{A}^n$. **Define** the linear polynomial $d_pF \in k[x_1, \dots, x_n]$ by
$$d_pF = dF|_{x=p} \cdot (x-p) = \sum_i \frac{\partial F}{\partial x_i}(p) \cdot (x_i - p_i).$$
- Let $X \subset \mathbb{A}^n$ be an *affine* variety with $\mathbb{I}(X) = (F_1, \dots, F_r)$. The **tangent space** T_pX **to** X **at** p is
$$T_pX = \mathbb{V}(d_pF_1, \dots, d_pF_r) = \bigcap_i \ker dF_i \subset \mathbb{A}^n.$$
- Note that $\mathbb{V}(d_pF_i)$ is a hyperplane, so T_pX is an intersection of hyperplanes, and thus a linear subvariety.
- A point $p \in X$ is a **smooth point** if $\dim_k T_pX = \dim_p X$.
- A point $p \in X$ is a **singular point** if $\dim_k T_pX > \dim_p X$. We write $\mathrm{Sing}(X)$ to be the collection of all singular points of X.
- *Let X be an* irreducible *affine variety of dimension d with $\mathbb{I}(X) = (F_1, \dots, F_r)$. Then $\mathrm{Sing}(X) \subset X$ is a closed subvariety given by the vanishing in X of all $(n-d) \times (n-d)$ minors of the Jacobian matrix*
$$\mathrm{Jac}(X) = \left(\frac{\partial F_i}{\partial x_j}\right)_{ij}$$
- *Let X be an affine variety and $p \in X$. Recall that $\mathfrak{m}_p = \{\frac{f}{g} \in \mathcal{O}_{X,p} \mid f(p) = 0\}$. Then there is a canonical vector-space isomorphism*
$$T_pX \cong (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$$
*where the vector space $\mathfrak{m}_p/\mathfrak{m}_p^2$ is called the **cotangent space**.*

  The proof is split into several steps:

  1. Prove for the case $X = \mathbb{A}^n$ and $p = 0$, using the fact that $\{d_0x_i = x_i\}$ gives a basis for $(T_0\mathbb{A}^n)^*$.
  2. Prove for general X with $p = 0$, after proving that $\overline{\mathfrak{m}}_p/\overline{\mathfrak{m}}_p^2 \cong \mathfrak{m}_p/\mathfrak{m}_p^2$, where $\overline{\mathfrak{m}}$ is the image of $\mathfrak{m}$ in the quotient $A(X) = R/\mathbb{I}(X)$.

  Fill in the details of the above proof.
- *T_pX depends only on an open neighbourhood of $p \in X$, and is thus an isomorphism invariant.*
- We define the **tangent space** of a *quasi-projective* variety X at a point $p \in X$ by $T_pX = (\mathfrak{m}_p/\mathfrak{m}_p^2)^*$. **However**, in practice we calculate the tangent space by picking an affine neighbourhood of $p \in X$ and then calculating the affine tangent space by using the Jacobian.

28.2 Derivative map

Let $F\colon X \to Y$ be a morphism of quasi-projective varieties.

- *Then $F^*\colon \mathcal{O}_{Y,F(p)} \to \mathcal{O}_{X,p}$ is a local (i.e. sends maximal ideals to maximal ideals) ring homomorphism $\mathfrak{m}_{F(p)} \to \mathfrak{m}_p$ given by $F^*g = g \circ F$.*

  This is simply because $g(F(p)) = 0 \implies (F^*g)(p) = 0$.
- We construct the **pullback map on cotangent spaces** $F^*\colon \mathfrak{m}_{F(p)}/\mathfrak{m}_{F(p)}^2 \to \mathfrak{m}_p/\mathfrak{m}_p^2$ by using the above claim: $F^*(\mathfrak{m}_{F(p)}) \subset \mathfrak{m}_p$ so $F^*(\mathfrak{m}_{F(p)}^2) \subset \mathfrak{m}_p^2$ and thus F^* is well defined on cotangent spaces.

- We define the **derivative map** $D_pF\colon T_pX \to T_{F(p)}X$ as the dual of the pullback map on cotangent spaces: $D_pF = (F^*)^*$.
- *On affine opens around p and $F(p)$ we can identify D_pF with the Jacobian matrix of F, i.e. locally $F\colon \mathbb{A}^m \to \mathbb{A}^n$ and $p = F(p) = 0$, and $\mathrm{Jac}(F) = \left(\frac{\partial F_i}{\partial x_j}\right)_{ij}$ acts by left multiplication $\mathbb{A}^m \equiv T_0\mathbb{A}^m \to \mathbb{A}^n \equiv T_0\mathbb{A}^n$.*

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29 Blow-ups

29.1 Blow-ups at a point

- The **blow-up $B_0\mathbb{A}^n$ of $\mathbb{A}^n$ at the origin** is the set of lines in $\mathbb{A}^n$ with a given choice of point:

$$B_0\mathbb{A}^n = \left\{(x,\ell) \in \mathbb{A}^n \times \mathbb{P}^{n-1} \mid x \in \ell\right\} = \mathbb{V}(x_iy_j - x_jy_i) \subset \mathbb{A}^n \times \mathbb{P}^{n-1}$$

  where we use coordinates $(x_1,\ldots,x_n)$ on $\mathbb{A}^n$ and $[y_1 : \ldots : y_n]$ on $\mathbb{P}^{n-1}$ (note that the projective coordinates are numbered *starting from* 1). Note that $x \in \ell$ means that $(x_1,\ldots,x_n) = \lambda(y_1,\ldots,y_n)$ for some $\lambda \in k^\times$.
- The **blow-up $B_p\mathbb{A}^n$ of $\mathbb{A}^n$ at a point** p is given by composing the linear isomorphism $x \mapsto x - p$ with the blow-up at the origin.
- The morphism $\pi\colon B_0\mathbb{A}^n \to \mathbb{A}^n$ **given by** $\pi\colon (x,[y]) \mapsto x$ is birational with inverse $x \mapsto (x,[x])$ defined on $x \neq 0$. The fibre $\pi^{-1}(x)$ is a point, apart from the **exceptional divisor** $E_0 = \pi^{-1}(0) = \{0\} \times \mathbb{P}^{n-1}$, which is a subvariety of codimension 1.
- *$\pi\colon B_0\mathbb{A}^n \setminus E_0 \to \mathbb{A}^n \setminus \{0\}$ is an isomorphism, and π collapses E_0 to the point 0.*
- *$E_0 \cong \mathbb{P}^{n-1} \cong \mathbb{P}(T_0\mathbb{A}^n)$ is the projectivisation of the tangent space: the closure of the preimage $\{(vt,[vt]) \mid t \neq 0\}$ of the punctured line $t \mapsto tv$ for $\neq 0$ contains the new point $(0,[v])$.*
- For an affine variety $X \subset \mathbb{A}^n$ *with* $0 \in X$ the **proper transform B_0X of** X is the closure

$$B_0X = \overline{\left(\pi^{-1}(X \setminus \{0\})\right)} \subset B_0\mathbb{A}^n.$$

- The **exceptional divisor** E **of** X is defined as $E = \pi^{-1}(0) \cap B_0X$.
- *$\pi\colon B_0X \to X$ is birational.*
- The **total transform of** X is $\pi^{-1}(X) = B_0X \cup E_0$.

29.2 Blow-ups along subvarieties and ideals

- Let X be an affine variety and $I = (f_1,\ldots,f_r) \lhd A(X)$. Define the **blow-up along an ideal** $B_I(X)$ to be the graph of $f\colon X \dashrightarrow \mathbb{P}^{r-1}$ where $f\colon x \mapsto [f_1(x) : \ldots : f_r(x)]$, i.e.

$$B_IX = \overline{\left\{(x,f(x) \mid x \in X \setminus \mathbb{V}(I)\right\}} \subset X \times \mathbb{P}^{r-1}.$$

- *$\pi\colon B_I(X) \to X$ given by $\pi\colon (x,[v]) \mapsto x$ is birational with inverse $x \mapsto (x,f(x))$ defined on $X \setminus \mathbb{V}(I)$.*
- The **exceptional divisor** E is $\pi^{-1}(\mathbb{V}(I))$.
- The **blow-up** B_YX **along an affine subvariety** $Y \subset X$ is defined as $B_{\mathbb{I}(Y)}X$.
- For a quasi-projective variety $X \subset \mathbb{P}^n$ and a homogeneous ideal $I \lhd S(\overline{X})$ with $I = (f_1,\ldots,f_r)$ we **define** $B_IX = B_I\overline{X} \cap (X \times \mathbb{P}^{r-1})$ by using that $f\colon \overline{X} \dashrightarrow \mathbb{P}^{r-1}$ determines $B_I\overline{X} \subset \overline{X} \times \mathbb{P}^{r-1}$ as before.

29.3 Examples

Work through lots of examples of blow-ups.

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30 Formalisation of quantum physics

30.1 Probability axioms

- Instead of working with probabilities (positive numbers) we work with **probability amplitudes** (complex numbers).
  - If independent events $E_1, \ldots, E_r$ with amplitudes z_i happen in sequence then we *multiply* amplitudes: $z = z_1 z_2 \ldots z_r$
  - If there are several possible ways $W_1, \ldots, W_s$ with amplitudes z_i of reaching the same outcome then we *add* amplitudes: $z = z_1 + z_2 + \ldots + z_s$.
- **Probability** is given by the *modulus of the square* of the amplitude: $p = |z|^2$.

30.2 Bra-ket notation

30.2.1 Definitions

- To an *isolated* physical system we associate a *complex Hilbert space* $\mathcal{H}$ *of dimension* n whose elements are **kets**, written as $|\bullet\rangle$ for any symbol $\bullet$. *We think of the dimension as telling us how many perfectly distinguishable configurations of our space there are.*
  - For example, $|\uparrow\rangle$ and $|\downarrow\rangle$ might be used for the kets associated to the spin of an electron.
  - The symbols on their own *have no meaning*: writing something like $\lambda|a\rangle = |\lambda a\rangle$ is an abuse of notation.
- The **inner product** of $\mathcal{H}$ is written as $\langle a|b\rangle$. *We think of it as telling us how well we can distinguish two states.*
- We define **bras** to be linear forms on $\mathcal{H}$, i.e. elements of the dual space $\mathcal{H}^*$, and they are written $\langle\bullet|$. The action of bras is defined by the inner product: $\langle a|(|b\rangle) = \langle a|b\rangle$.
- We also have the **outer product** $|a\rangle\langle b|$ which is a linear operator whose action is defined by $|a\rangle\langle b|(|c\rangle) = |a\rangle\langle b|c\rangle$, i.e. we scale $|a\rangle$ by $\langle b|-\rangle$.
- Given some state $|\Psi\rangle$ the outer product $|\Psi\rangle\langle\Psi|$ is called the **projector on** Ψ.
- Given some linear operator A we can write $\langle a|A|b\rangle$ unambiguously, since $\langle a|\Big(A|b\rangle\Big)$ and $\Big(\langle a|A\Big)|b\rangle$ are equal (this is made clear when we talk about *state vectors*).

30.2.2 Useful identities

1. $\Big(\alpha_1|\psi_1\rangle + \alpha_2|\psi_2\rangle\Big)^\dagger = \alpha_1^*\langle\psi_1| + \alpha_2^*\langle\psi_2|$
2. $\langle\phi|\psi\rangle^\dagger = \langle\psi|\phi\rangle$
3. $\Big(\langle\phi|AB|\psi\rangle\Big)^\dagger = \langle\psi|B^\dagger A^\dagger|\phi\rangle$
4. $\Big(\alpha_1|\psi_1\rangle\langle\phi_1| + \alpha_2|\psi_2\rangle\langle\phi_2|\Big)^\dagger = \alpha_1^*|\phi_1\rangle\langle\psi_1| + \alpha_2^*|\phi_2\rangle\langle\psi_2|$

30.3 State vectors

- We usually assume that $\mathcal{H}$ has an *orthonormal basis* $\{|e_1\rangle, \ldots, |e_n\rangle\}$.
  - We can think of a ket $|a\rangle = \sum_i \alpha_i|e_i\rangle$ as a column vector, where $\sum_i |\alpha_i|^2 = 1$.
  - We can think of a bra $\langle b| = \sum_i \beta_i\langle e_i|$ a a row vector, where $\sum_i |\beta_i|^2 = 1$.
  - The **Hermitian conjugate** $\dagger$ is defined as the *complex conjugation of the matrix transpose*:

$$\begin{pmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{pmatrix}^\dagger = (\alpha_1^* \ \ldots \ \alpha_n^*)$$

    where z^* denotes the complex conjugate of z. That is, *the Hermitian conjugate turns a ket into the corresponding dual bra, and vice versa.*

  - We can think of linear operators as $n \times n$ matrices over $\mathbb{C}$, and then $\langle a|A$ and $A|b\rangle$ both make sense, using matrix multiplication, and $\langle a|\Big(A|b\rangle\Big) = \Big(\langle a|A\Big)|b\rangle$.

- We always assume that state vectors have been **normalised**, i.e. are of length 1.
- Any two state vectors that differ by a **global phase** describe the same quantum state, i.e. $|\Psi\rangle$ and $e^{i\theta}|\Psi\rangle$ are 'the same'.
- But **relative phase** factors are very important, i.e. $\alpha_0|0\rangle+\alpha_1|1\rangle$ and $\alpha_0|0\rangle+e^{i\theta}\alpha_1|1\rangle$ are two *very different* states (for $\theta \neq 2m\pi$).
- If a state is of the form $|\Psi\rangle \in \mathcal{H}$ then we say it is a **pure state**, i.e. there is no probability involved in what the state is – it *is* in state $|\Psi\rangle$. If we know that e.g. the state is $|\Psi\rangle$ with probability $\frac{1}{2}$ and state $|\Psi'\rangle$ with probability $\frac{1}{2}$ then we call this a **mixed state**.
- If a state is of the form $\alpha_1|e_1\rangle + \ldots + \alpha_n|e_n\rangle$ with $\sum_i \alpha_i = 1$ and at least *two* α_i are non-zero then we say that the state is a **(coherent) superposition**. Note that a superposition is *still a pure state, i.e. not a mixed state*. (This is better explained with the language of *density operators*.)

30.4 Qubits

- A **quantum bit**, or **qubit**, is a two-dimensional complex Hilbert space.
- We often describe the state of a qubit in terms of the orthonormal basis $\{|0\rangle, |1\rangle\}$, i.e. $\alpha_0|0\rangle + \alpha_1|1\rangle$ with $|\alpha_0|^2 + |\alpha_1|^2 = 1$. This basis is called the **computational basis**.
- Working in this basis, we have the **projectors** P_0, P_1 which are linear operators defined by $P_i = |i\rangle\langle i|$.
- The probability of a state $|\Psi\rangle$ being measured and resulting in $|i\rangle$ is given by $\langle\Psi|P_0|\Psi\rangle$.

30.5 Entanglement

- If we have two quantum systems A and B with corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ then we describe elements of the combined space $\mathcal{H}_{A,B}$ with the *tensor product of Hilbert spaces* $\mathcal{H}_A \otimes \mathcal{H}_B$. That is, elements are *linear combinations* of elementary tensors $|\Psi_A\rangle \otimes |\Psi_B\rangle$.
- There are various ways of writing tensor products of pure states, and we use any of the following:
  - $|\Psi_A\rangle \otimes |\Psi_B\rangle$
  - $|\Psi_A\rangle|\Psi_B\rangle$
  - $|\Psi_A\Psi_B\rangle$
- Given some element $|\Psi_A\Psi_B\rangle$ the corresponding bra is $\langle\Psi_A\Psi_B|$, and these act coordinate-wise, i.e.

$$\langle\Phi_A\Phi_B|\Big(|\Psi_A\Psi_B\rangle\Big) = \Big(\langle\Phi_A|\otimes\langle\Phi_B|\Big)\Big(|\Psi_A\rangle\otimes|\Psi_B\rangle\Big) = \Big(\langle\Phi_A|\Psi_A\rangle\Big)\otimes\Big(\langle\Phi_B|\Psi_B\rangle\Big).$$

- Note that elements of $\mathcal{H}_A \otimes \mathcal{H}_B$ are *linear combinations*. That is, not all elements in $\mathcal{H}_A \otimes \mathcal{H}_B$ are of the form $|\Psi_A\rangle \otimes |\Psi_B\rangle$ for $|\Psi_i\rangle \in \mathcal{H}_i$. In fact, *most* elements *aren't* of this form.
  - If $|\Psi\rangle \in \mathcal{H}_{A,B}$ can be written as $|\Psi_A\rangle \otimes |\Psi_B\rangle$ then we say that it is a **separable state**, e.g. $\alpha|00\rangle + \beta|01\rangle = |0\rangle \otimes (\alpha|0\rangle + \beta|1\rangle)$
  - If $|\Psi\rangle \in \mathcal{H}_{A,B}$ is *not* separable then we say that it is an **entangled state**, e.g. $\alpha|00\rangle + \beta|11\rangle \neq |\Psi_A\rangle \otimes |\Psi_B\rangle$ for any $|\Psi_i\rangle \in \mathcal{H}_i$.
- Working in the computational basis $\{|0\rangle, |1\rangle\}$, we can define an **quantum n-register** to be the tensor product of n qubits, whose elements are *linear combinations* of $|x\rangle$ where $x \in \{0,1\}^n$.

30.6 Unitary evolutions

- A linear operator U is said to be **unitary** if $U^\dagger U = UU^\dagger = \mathbb{1}$.
- In our simple formalisation of quantum physics, **all evolutions are unitary**. This is because unitary operators are exactly what we need to ensure that state vectors evolve into state vectors.
- If we start with some state $|\Psi\rangle$ and let it evolve under unitary transforms $U_1, \ldots, U_r$ then the final state will be $U_r U_{r-1} \ldots U_1 |\Psi\rangle$, i.e. *composition corresponds to matrix multiplication*.
- Given two spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and two unitary operations U_A and U_B, we can define the **tensor-product operator** $U_A \otimes U_B$ that acts on $\mathcal{H}_A \otimes \mathcal{H}_B$ by the *matrix outer product*: write U_A and U_B as matrices $U_A = (a_{ij})_{ij}$ and $U_B = (b_{ij})_{ij}$ with $\dim \mathcal{H}_A = m$ and $\dim \mathcal{H}_B = n$. Then

$$U_A \otimes U_B = \begin{pmatrix} a_{11}U_B & a_{12}U_B & \ldots & a_{1m}U_B \\ a_{21}U_B & a_{22}U_B & \ldots & a_{2m}U_B \\ \vdots & & \ddots & \vdots \\ a_{m1}U_B & a_{22}U_B & \ldots & a_{mm}U_B \end{pmatrix} \in \mathsf{Mat}_{mn\times mn}(\mathbb{C}).$$

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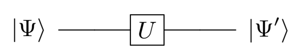
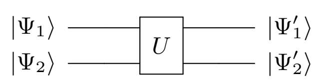
31 Introduction to quantum circuits

31.1 Quantum circuit diagrams

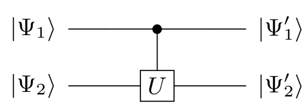
- Say that we start with a state $|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ and let it undergo a unitary evolution U, so that we end with state $|\Psi'\rangle = U|\Psi\rangle$. We write this as a **quantum circuit** as follows:



- If we start with an entangled state $|\Psi\rangle = |\Psi_1\rangle|\Psi_2\rangle$ and let it undergo a unitary evolution U then we write this as follows:



- A **controlled** U**-gate** is used to represent an operation on an entangled state where one subsystem affects the other, i.e. the second qubit evolves under U if the first qubit is in a certain state, and remains the same if the first qubit is in a different state. (This is explained in more detail further on.) These are written as follows:



31.2 Single-qubit gates

For all that follows we work in the computational basis $\{|0\rangle, |1\rangle\}$.

- The **square-root-of-**NOT **gate** $\sqrt{\text{NOT}}$ is described by the matrix

$$\sqrt{\text{NOT}} = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} \\ \frac{i}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

- The φ**-phase gate** P_φ is described by the matrix

$$P_\varphi = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\varphi} \end{pmatrix}$$

  for some $\varphi \in (0, 2\pi]$.

- The **Hadamard gate** H is described by the matrix

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

- The **Pauli gates** $\sigma_{(-)}$ are described by the matrices

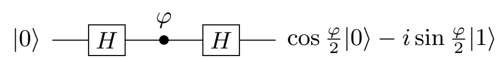
$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \text{NOT}$$

$$\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = i\sigma_x\sigma_z$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = P_\pi$$

31.3 The fundamental single-qubit circuit

The building block of all quantum circuits is the following circuit:



31.4 Multi-qubit gates

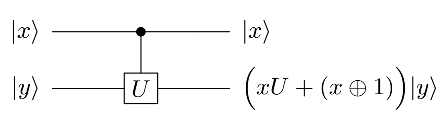
- Often we work with a two-qubit system and the computational basis $|e_i\rangle$ where $e_i \in \{0,1\}^2$, and ordered as follows: $|00\rangle, |01\rangle, |10\rangle, |11\rangle$. We also often assume that our first qubit (our input qubit) is in state $|0\rangle$.
- Given two n-registers $x, y \in \{0,1\}^n$ we define their **dot product** $x \cdot y \in \{0,1\}$ as the usual dot product (with addition $\oplus$ modulo 2), i.e. writing $x = x_1x_2\ldots x_n$ and $y = y_1y_2\ldots y_n$ we have $x \cdot y = x_1y_1 \oplus x_2y_2 \oplus \ldots \oplus x_ny_n$.
- For $x \in \{0,1\}^n$ the **Hadamard transform** is *also* given by

$$|x\rangle \mapsto \frac{1}{\sqrt{2}} \sum_{y \in \{0,1\}^n} (-1)^{x \cdot y} |y\rangle.$$

- Given a unitary transformation U acting on a qubit, we define the **controlled-U gate** as the gate acting on a 2-register by applying U to the second qubit if the first qubit is a 1, and applying the identity to the second qubit if the first qubit is a 0. This is written as a matrix as follows:

$$C_U = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & U_{11} & U_{12} \\ 0 & 0 & U_{21} & U_{22} \end{pmatrix}$$

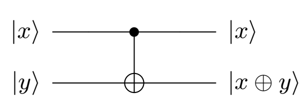
and written in a circuit as • for the control qubit and U for the operand qubit. When $x, y \in \{0,1\}$, it acts as follows:



- A specific example of a controlled-U gate is the **controlled-NOT gate**, given by the matrix

$$C_{\text{NOT}} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

and written in a circuit as $\oplus$. When $x, y \in \{0,1\}$, it acts as follows:



QI 32 No-cloning and teleportation

32.1 No-cloning

Suppose that we have some circuit C that clones the input, i.e. for any state $|\Psi\rangle$, and writing $|e\rangle$ to mean the state of the environment, the machine acts as follows

$$|\Psi\rangle|0\rangle|e\rangle \;-\!\boxed{C}\!-\; |\Psi\rangle|\Psi\rangle|e'\rangle$$

If we pick states $|\Psi\rangle$ and $|\Phi\rangle$ that are *non-orthogonal* and *non-identical*, i.e. $\langle\Phi|\Psi\rangle \notin \{0,1\}$, then running the cloning machine gives us outputs

$$|\Psi\rangle|0\rangle|e\rangle \;-\!\boxed{C}\!-\; |\Psi\rangle|\Psi\rangle|e'\rangle$$

$$|\Phi\rangle|0\rangle|e\rangle \;-\!\boxed{C}\!-\; |\Phi\rangle|\Phi\rangle|e''\rangle$$

But since the evolution must be unitary, it preserves the inner product, and so

$$\langle\Phi|\langle 0|\langle e|\Big(|\Psi\rangle|0\rangle|e\rangle\Big) = \langle\Phi|\langle\Phi|\langle e''|\Big(|\Psi\rangle|\Psi\rangle|e'\rangle\Big)$$

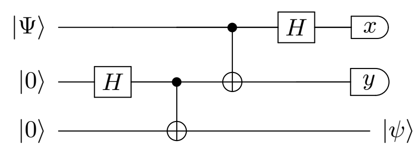
which is satisfied iff

$$\langle\Phi|\Psi\rangle = \langle\Phi|\Psi\rangle^2\langle e'|e''\rangle.$$

This last equality requires that $\langle\Phi|\Psi\rangle \in \{0,1\}$, which contradicts our initial assumptions, and so we see that no such cloning circuit C can exist.

32.2 Teleportation

Suppose we have some state $|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ that we wish to teleport. We already know that we can't clone it, so if we *can* teleport it then we must end up destroying our original state at some point before the receiver has it. The following circuit provides **teleportation**, where Alice has access to the *first two qubits*, and Bob has access to the *third qubit*:



The circuit is run, *and then Alice broadcasts the measurements of* $|x\rangle$ *and* $|y\rangle$. To recover the initial state $|\Psi\rangle$, Bob has to *modify* $|\psi\rangle$ *in a way that depends on the measurement of the ancillary qubits* $|x\rangle|y\rangle$ according to the following table:

$\lvert xy\rangle$	$\lvert\Psi\rangle$
$\lvert 00\rangle$	$\lvert\psi\rangle$
$\lvert 01\rangle$	$\sigma_x\lvert\psi\rangle$
$\lvert 10\rangle$	$\sigma_z\lvert\psi\rangle$
$\lvert 11\rangle$	$\sigma_z\sigma_x\lvert\psi\rangle$

QI 33 Quantum interference and decoherence

- The fundamental quantum circuit is the following:

$$|\Psi\rangle \;-\!\boxed{H}\!-\!\overset{\varphi}{\bullet}\!-\!\boxed{H}\!-\; HP_\varphi H|\Psi\rangle$$

- We know how this acts in a *closed system*, and we can represent it by the following matrix:

$$HP_\varphi H = \begin{pmatrix} \cos\frac{\varphi}{2} & -i\sin\frac{\varphi}{2} \\ -i\sin\frac{\varphi}{2} & \cos\frac{\varphi}{2} \end{pmatrix}$$

- If we input $|\Psi\rangle = |0\rangle$ then the final state will be $|\Psi'\rangle = \cos\frac{\varphi}{2}|0\rangle - i\sin\frac{\varphi}{2}|1\rangle$. Then the probability p_0 of measuring the final state *and seeing* $|0\rangle$ is

$$\langle\Psi'|P_0|\Psi'\rangle = \langle\Psi'|0\rangle\langle 0|\Psi'\rangle = \cos^2\frac{\varphi}{2} = \frac{1}{2}(1+\cos\varphi).$$

- In reality we have the **ambient environment** $|e\rangle$, i.e. **decoherence** D happens somewhere along the circuit (we assume that it happens after the phase), and we formalise this as follows:
  1. $|\Psi\rangle = |\Psi\rangle|e\rangle$
  2. $|0\rangle|e\rangle \xrightarrow{D} |0\rangle|e_0\rangle$
  3. $|1\rangle|e\rangle \xrightarrow{D} |1\rangle|e_1\rangle$
  4. $\langle e_0|e_1\rangle = ve^{i\alpha}$ for some **visibility** $v \in [0,1]$ and **phase** $\alpha \in (0, 2\pi]$.

  Note that $|e_0\rangle, |e_1\rangle$ are not necessarily orthogonal, but *are assumed to be of unit length*.
- Under these new assumptions we can see how our fundamental circuit acts on $|0\rangle$:

$$|0\rangle|e\rangle \xrightarrow{HP_\varphi H} |\Psi'\rangle = \frac{1}{2}\Big(|e_0\rangle + e^{i\varphi}|e_1\rangle\Big)|0\rangle + \frac{1}{2}\Big(|e_0\rangle - e^{i\varphi}|1\rangle\Big)|1\rangle.$$

- The **projection operators** P_i are defined as $|i\rangle\langle i| \otimes \mathbb{1}$, i.e. they act as the identity on the environment.
- We can check that $P_i^2 = P_i$ and $P_0P_1 = 0$. Thus the P_i are **orthogonal projections**.
- We can calculate

$$\langle\Psi'|P_0|\Psi'\rangle = \frac{1}{2}\Big(1 + v\cos(\varphi+\alpha)\Big).$$

- **Interpretation:**
  1. If $\alpha = 0$ then we are just reducing sensitivity to φ, and this is controlled by v:
     (a) if $v = 0$ then the environment 'knows' exactly what is happening, and we see that $p_0 = p_1 = \frac{1}{2}$, i.e. all outcomes are equally likely;
     (b) if $v = 1$ then the environment can't distinguish between $|e_0\rangle$ and $|e_1\rangle$, and we we recover the original probability of $p_0 = \frac{1}{2}(1+\cos\varphi)$.
  2. If $\alpha \neq 0$ then this means we can't really predict anything about the outcome, *unless we know* α, in which case we can fully account for this.

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34 Entanglement and controlled-unitary gates

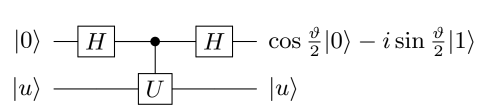
- Given some unitary operator U on an n-register, we can define a **controlled-U gate** on an $n+1$ register by defining

$$\begin{aligned}|0\rangle|v\rangle &\to |0\rangle\mathbb{1}|v\rangle\\ |1\rangle|v\rangle &\to |1\rangle U|v\rangle\end{aligned}$$

  i.e. the first qubit acts as a control bit.
- If we let $|u\rangle$ be some *eigenstate* of U, i.e. $U|u\rangle = e^{i\vartheta}|u\rangle$ for some $\vartheta \in (0, 2\pi]$, then the controlled-U gate acts as follows:

$$\begin{aligned}|0\rangle|v\rangle &\to |0\rangle|v\rangle\\ |1\rangle|v\rangle &\to e^{i\vartheta}|1\rangle|v\rangle\end{aligned}$$

  i.e. on the *computational basis*, the controlled-U gate acts as the identity on all of the register (since we can ignore global phase).
- Generalising the fundamental circuit (Hadamard-phase-Hadamard) we obtain the following **controlled-U circuit**:



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35 Quantum algorithms

35.1 Boolean functions and oracles

- A **boolean function** is a function $f\colon \{0,1\}^m \to \{0,1\}^n$.
- An **oracle** is a 'black box', i.e. a circuit into which we feed an input and receive an output, but without knowing anything about how the circuit works.
- Given some boolean function $f\colon \{0,1\}^m \to \{0,1\}^n$ we define the **associated controlled-U gate**, which acts on an $(m+n)$-register, as follows (where $x \in \{0,1\}^m, y \in \{0,1\}^n$):

$$|x\rangle|y\rangle \xrightarrow{f} |x\rangle|y \oplus f(x)\rangle.$$

(We might think of simply using $|x\rangle \to |f(x)\rangle$, but note that this is *not* unitary, i.e. reversible.)

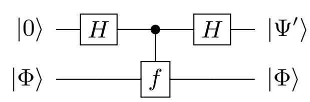
35.2 Deutch's algorithm

- **Deutch's algorithm** is constructed to answer the following question, which deals with the case $m = n = 1$:

Suppose we have some oracle that encodes a controlled-U gate, where U is given by a boolean function $f\colon \{0,1\} \to \{0,1\}$, so that f is either constant (i.e. always outputs the same value), or balanced (i.e. outputs either value with equal probability). How many calls to the oracle must we make to determine whether f is constant or balanced?

The *classical* answer is *two* calls; the *quantum* answer, using Deutch's algorithm, is *one* call.

- The algorithm relies on a clever choice of the second qubit, namely the eigenstate $|\Phi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$ of f, and two Hadamard gates sandwiching the controlled-U, i.e. we run the circuit



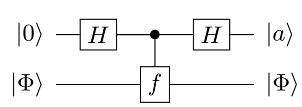
noting that $|\Phi\rangle$ is an eigenstate of f.

Then

$$\begin{aligned}|0\rangle|\Phi\rangle &\xrightarrow{H\otimes 1} \sum_{x\in\{0,1\}} |x\rangle|\Phi\rangle \\ &\xrightarrow{f} \sum_{x\in\{0,1\}} |x\rangle\Big(|f(x)\rangle - |f(x)\oplus 1\rangle\Big) = \sum_{x\in\{0,1\}} (-1)^{f(x)}|x\rangle|\Phi\rangle \\ &\xrightarrow{H\otimes 1} \begin{cases}|0\rangle & \text{if } f \text{ constant;}\\ |1\rangle & \text{if } f \text{ balanced.}\end{cases}\end{aligned}$$

35.3 Bernstein-Vazirani problem

- The **Bernstein-Vazirani problem** deals with the specific case when $n = 1$ and f is of the form $f(x) = x \cdot a = x_1 a_1 \oplus \ldots \oplus x_m a_m$ for some $a \in \{0,1\}^m$.
- The *classical* amount of calls needed to determine f is n; the *quantum* amount of calls needed is just *one*. Setting $|\Phi\rangle = \frac{1}{\sqrt{2}}\{|0\rangle - |1\rangle\}$, which is an eigenstate of f, we run the following circuit:



- To see that we do in fact recover a from this circuit, consider the following computation:

$$\begin{aligned}|0\rangle|\Phi\rangle &\xrightarrow{H\otimes\mathbb{1}} \sum_{x\in\{0,1\}^m}|x\rangle|\Phi\rangle \\ &\xrightarrow{f} \sum_{x\in\{0,1\}^m}(-1)^{x\cdot a}|x\rangle|\Phi\rangle \\ &\xrightarrow{H\otimes\mathbb{1}} \sum_{x'\in\{0,1\}^m}\left(\sum_{x\in\{0,1\}^m}(-1)^{x\cdot(a\oplus x')}\right)|x'\rangle|\Phi\rangle = |a\rangle|\Phi\rangle.\end{aligned}$$

  We justify the last equality as follows: summing over all $x \in \{0,1\}^m$ means that the $(-1)^k$ terms will all cancel in pairs *except* for when $a \oplus x' = 0$ which happens iff $x' = a$.

QI 36 Bell inequalities

36.1 The setup

- Consider the following experiment:
  - Alice and Bob each receive a qubit that form an entangled pair.
  - There are *two* measurements, A_1, A_2 for Alice and B_1, B_2 for Bob, that they can each perform on their qubit, and *both* are encoded in the computational basis. For example, A_i measures spin along some direction α_i and B_i measures spin along some direction β_i, and we associate spin 1 with $|1\rangle$ and spin -1 with $|0\rangle$.
  - They then repeat the following two steps many many times:
    1. Alice and Bob both flip a fair coin to decide which measurement they will perform (i.e. they decide *independently from each other* and *randomly*).
    2. They perform their chosen measurement and then share their results with each other.
- There are now three ways of formulating some sort of **Bell inequality**:
  1. We can think of the A_i and B_i as *random variables* taking the values ± 1. We thus define a new random variable

     $$S = A_1(B_1 - B_2) + A_2(B_1 + B_2).$$

     Now thinking of the A_i and B_i as random variables (of which we have some statistical knowledge from repeating the above steps), we see that $S = \pm 2$ and thus

     $$-2 \leqslant \mathbb{E}(S) \leqslant 2.$$

     This is one of the **Bell inequalities**.
  2. It might be that there is some sort of correlation between the measurements, depending on how exactly the qubits are prepared. Assume that we have the following correlation:

     $$A_1 = B_1 = A_2 \neq B_2.$$

     That is, whenever Alice chooses to measure A_1 and Bob chooses to measure B_2 they satisfy $A_1 \neq B_2$, etc. There are three very important things to note here:
     - these correlations are *globally inconsistent*, i.e. we cannot choose values for A_1, A_2, B_1, and B_2 such that all the correlations are satisfied at once: we can satisfy *at most three* of them; *but* ...
     - ... *this doesn't mean that these correlations cannot physically occur*, simply because Alice and Bob only ever perform *one* measurement each time, and the correlations are definitely pairwise satisfiable; *but* ...

– . . . *we don't know of any way to physically realise these correlations.*

As mentioned, *if we choose predetermined values* for the A_i and the B_i then we cannot simultaneously satisfy all of the correlations. There are two ways of expressing this as a **Bell inequality**, which tell us how far away from perfectly satisfying these correlations we are *with our predetermined values*:

(a) Define $I_2 = \mathbb{P}(A_1 \neq B_1) + \mathbb{P}(B_1 \neq A_2) + \mathbb{P}(A_2 \neq B_2) + \mathbb{P}(A_1 = B_2)$. Since for any given A_i, B_i at least one of the correlations won't hold, we see that

$$I_2 \geqslant 1.$$

(b) Let R be the event that *just one* randomly chosen correlation (e.g. $A_2 = B_1$) holds. Then

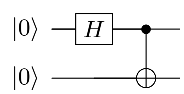
$$\mathbb{P}(R) \geqslant 0.25.$$

36.2 The quantum violation

- We claim that, with entanglement, *we can violate the Bell inequalities*. In particular, we can achieve
  1. $\mathbb{E}(S) = 2\sqrt{2} \approx 2.8$;
  2. (a) $I_2 = 4\sin^2 \frac{\pi}{8} \approx 0.6$;
     (b) $\mathbb{P}(R) = \sin^2 \frac{\pi}{8} \approx 0.15$.
- To see this we use the entangled state

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$$

where the qubits are *polarised photons*, and the measurements A_i are of polarisation along some angle α_i and the B_i are of polarisation along some angle β_i, where we will choose α_i and β_i 'cleverly'. We can construct this state (the **Bell state**) by using the following circuit:



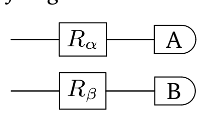
$\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$

- A physical fact (which doesn't really concern us too much) is that measuring polarisation along some angle φ is equivalent to rotating our qubits by φ and then measuring in the computational basis, where the **rotation operator** R_φ is given by the matrix

$$R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi \\ \sin\varphi & \cos\varphi \end{pmatrix}$$

- We first consider the simple case, where $A_i = A$ and $B_i = B$, i.e. Alice and Bob *don't have a choice of angle*. This is described by the following circuit:

$\frac{1}{\sqrt{2}}\Big(|00\rangle + |11\rangle\Big)$



$\cos(\beta - \alpha)\frac{|00\rangle + |11\rangle}{\sqrt{2}} + \sin(\beta - \alpha)\frac{|01\rangle + |10\rangle}{\sqrt{2}}$

whose output state is such that

$$\mathbb{P}(A = B) = \cos^2(\beta - \alpha)$$
$$\mathbb{P}(A \neq B) = \sin^2(\beta - \alpha)$$

- We use the following 'clever' choice of angles (it turns out that these actually give us the *strongest lower bound possible within the realms of quantum physics*):

$$\alpha_1 = 0 \qquad \beta_1 = \frac{\pi}{8}$$
$$\alpha_2 = \frac{2\pi}{8} \qquad \beta_2 = \frac{3\pi}{8}$$

and recall that A_i is the measurement of polarisation along α_i by Alice, and B_i is the measurement of polarisation along β_i by Bob.

- Combining all of the above, we see that

$$\mathbb{P}(A_1 = B_1) = \mathbb{P}(B_1 = A_2) = \mathbb{P}(A_2 = B_2) = \mathbb{P}(B_2 \neq A_1) = \cos^2 \frac{\pi}{8}$$

and we thus violate two of the Bell inequalities:
(a) $I_2 = 4\sin^2 \frac{\pi}{8} \approx 0.6$;
(b) $\mathbb{P}(R) = \sin^2 \frac{\pi}{8} \approx 0.15$.
- For the remaining Bell inequality we return first to the simple case: note that if $A = B$ then $AB = 1$ and if $A \neq B$ then $AB = -1$. Thus

$$\begin{aligned}\mathbb{E}(AB) &= \cos^2(\beta - \alpha) - \sin^2(\beta - \alpha) \\ &= \cos^2(\beta - \alpha).\end{aligned}$$

So, returning to our previous choices of α_i, β_i, we see that

$$\begin{aligned}\mathbb{E}(S) &= \mathbb{E}(A_1B_1) - \mathbb{E}(A_1B_2) + \mathbb{E}(A_2B_1) + \mathbb{E}(A_2B_2) \\ &= 2\sqrt{2}\end{aligned}$$

which violates the last Bell inequality.

QI

37 Density operators

37.1 Preliminary definitions

- There are a few problems with using state vectors to describe quantum physics that density operators aim to solve, with the last point being the most important:
  1. two states can differ by a global phase but still be experimentally the same;
  2. we generally have a lot of tensor products floating around;
  3. state vectors can only really describe *pure states*, and not *mixed states*.
- An operator A on $\mathcal{H}$ is **positive semi-definite** if $\langle\Psi|A|\Psi\rangle \geqslant 0$ for all $|\Psi\rangle \in \mathcal{H}$.
- Say we have some system (isolated or not, and possibly a tensor product) $\mathcal{H}$. Then we describe this system by **density operators**, which are defined to be positive semi-definite linear operators of trace 1.
- *A positive semi-definite linear operator of trace* 1 *is Hermitian.*

  Make sure that you can prove this.
- $\operatorname{tr}|a\rangle\langle b| = \langle b|a\rangle$

  Make sure that you can prove this too.

37.2 Pure states

- To a *pure state* $|\Psi\rangle \in \mathcal{H}$ we associate the **density operator** $\rho = |\Psi\rangle\langle\Psi|$.
- Say $|\Psi\rangle \in \mathcal{H}$ is a pure state. Then working in the computational basis we can write the associated density operator ρ as a **density matrix**:

$$\begin{aligned}|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle \leftrightarrow \rho &= |\alpha_0|^2|0\rangle\langle 0| + \alpha_0\alpha_1^*|0\rangle\langle 1| + \alpha_0^*\alpha_1|1\rangle\langle 0| + |\alpha_1|^2|1\rangle\langle 1| \\ &= \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \\ \alpha_0^*\alpha_1 & |\alpha_1|^2 \end{pmatrix}\end{aligned}$$

- The diagonal elements of the density matrix are called **populations**; the off-diagonal elements are called **coherences**.
- Recall that we have projectors $P_i = |i\rangle\langle i|$ that satisfy $P_iP_j = \delta_{ij}$ and $P_0 + P_1 = \mathbb{1}$, and that the probability of measuring $|\Psi\rangle$ to be in state $|i\rangle$ is given by $\langle\Psi|P_i|\Psi\rangle$. We extend this to define **probabilities for density operators** by noting that

$$\langle\Psi|P_i|\Psi\rangle = \operatorname{tr} P_i|\Psi\rangle\langle\Psi| = \operatorname{tr} P_i\rho$$

and thus set the **probability of measuring ρ to be in state $|i\rangle$** as $\operatorname{tr} P_i\rho$.

Make sure that you can prove the first equality above.

37.3 Mixed states

- Recall that a mixed state is a statistical ensemble of quantum states, i.e. we know that the system is in state $|\Psi_k\rangle$ with probability p_k. To such a mixed state we associate the **density operator**

$$\rho = \sum_k p_k |\Psi_k\rangle\langle\Psi_k|.$$

- The **probability of measuring a mixed state ρ to be in the state $|i\rangle$** is given by

$$\sum_k \langle\Psi_k|P_i|\Psi_k\rangle = \operatorname{tr} \sum_k p_k P_i |\Psi_k\rangle\langle\Psi_k| = \operatorname{tr} P_i\rho.$$

Make sure that you can prove the first equality above.

37.4 Experimental differentiability

- *By definition, density operators and projectors commute. Thus*

$$\mathbb{P}\big(\rho \text{ is measured to be in state } |i\rangle\big) = \operatorname{tr} \rho P_i = \operatorname{tr} P_i\rho.$$

- *A mixed state has zero coherence terms; a pure state has maximal coherence terms* $|\rho_{ij}| = \sqrt{\rho_{ii}\rho_{jj}}$.

  Try to prove this.

- *A density operator ρ describes a pure state iff* $\operatorname{tr} \rho^2 = 1$.

  Try to prove this (assuming the above).

- We now describe two states, one pure and one mixed, that we *cannot* experimentally differentiate simply by measuring in the computational basis. (Though it is important to note that we *can* differentiate them by measuring in some other basis.)

  For both of the following states, the probability of measurement resulting in $|0\rangle$ is $|\alpha_0|^2$ and the probability of measurement resulting in $|1\rangle$ is $|\alpha_1|^2$.

  1. Say we have a qubit that is in state $|0\rangle$ with probability α_0 and in state $|1\rangle$ with probability α_1. Then the associated density matrix ρ is given by

$$\rho = \begin{pmatrix} |\alpha_0|^2 & 0 \\ 0 & |\alpha_1|^2 \end{pmatrix}$$

     (and this describes a *mixed state*).

  2. Say we have a qubit in the coherent superposition $\alpha_0|0\rangle + \alpha_1|1\rangle$. We have already seen that this has density matrix ρ given by

$$\rho = \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \\ \alpha_0^*\alpha_1 & |\alpha_1|^2 \end{pmatrix}$$

     (which describes a *pure state*).

  We know that there is some way of differentiating these states experimentally since they are *represented by different density matrices*.

- Converse to the above example, we now describe two (mixed) states which give rise to the same density matrix, and thus *cannot* be experimentally differentiated, *no matter which basis we measure in*.

  1. Say we have a qubit that is in states $|0\rangle$ and $|1\rangle$ with equal probability $\frac{1}{2}$. Then this is the mixed state in the above example with $\alpha = \beta = \frac{1}{\sqrt{2}}$ and so the density matrix is

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

2. Say we have a qubit that is in the states $\frac{1}{\sqrt{2}}\big(|0\rangle + |1\rangle\big)$ and $\frac{1}{\sqrt{2}}\big(|0\rangle - |1\rangle\big)$ with equal probability $\frac{1}{2}$. Then this mixed state also has density matrix ρ given by

$$\rho = \begin{pmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{pmatrix}$$

- **Motto:** density operators describe states *up to experimental differentiability*.

37.5 Partial trace and the environment

- The **partial trace of an operator over a subsystem** is defined for operators of the form $A = \bigotimes_{i=1}^{k} A_i$ on a system of the form $\mathcal{H} = \bigotimes_{i=1}^{k} \mathcal{H}_{A_i}$ as follows:

$$\mathrm{tr}_{A_i} A = (\mathrm{tr}\, A_i) \bigotimes_{i \neq j} A_j$$

  and is extended linearly to all operators on $\mathcal{H}$.
- *The partial trace is the only function f such that*

$$tr[P_k f(\rho)] = \mathrm{tr}[(P_k \otimes \mathbb{1})\rho].$$

- Partial traces give a way of *disregarding certain subsystems*. For example, say that we have two entangled subsystems $\mathcal{H}_A$ and $\mathcal{H}_B$ with orthonormal basis states $|a_i\rangle|b_i\rangle$, and define the mixed state

$$|\Psi\rangle = \sum_k \sqrt{p_k}\, |\Psi_k\rangle|b_k\rangle$$

  where $p_k \geqslant 0$ are such that $\sum_k p_k = 1$. This state has associated density operator ρ given by

$$\rho = |\Psi\rangle\langle\Psi| = \sum_{k,\ell} \sqrt{p_k p_\ell}\, \Big(|\Psi_k\rangle\langle\Psi_\ell| \otimes |b_k\rangle\langle b_\ell|\Big).$$

  Tracing over the subsystem B then gives

$$\mathrm{tr}_B\, \rho = \sum_{k,\ell} \Big(\langle b_\ell|b_k\rangle \sqrt{p_k p_\ell}\Big)\, |\Psi_k\rangle\langle\Psi_\ell| = \sum_k p_k |\Psi_k\rangle\langle\Psi_k| = \rho_A$$

  where ρ_A is the density operator associated to system A. This is exactly the same as our previous scenarios, i.e. *tracing over a subsystem is a 'valid' operation*.
- **Interpretation:** when we describe mixed states as being 'state $|\Psi_k\rangle$ with probability p_k' and thus described by density operator $\sum_k p_k|\Psi_k\rangle\langle\Psi_k|$ this is really just an educated guess. The actual prepared state is of the form $\sum_{k,\ell} \sqrt{p_k p_\ell}\, \Big(|\Psi_k\rangle\langle\Psi_\ell| \otimes |b_k\rangle\langle b_\ell|\Big)$ where $\mathcal{H}_B$ corresponds to the universe and the $|b_k\rangle$ correspond to the universe knowing things that we don't. We recover our educated guess by 'tracing over the universe'.

37.6 Decoherence

37.6.1 The general case

- Recall the notation from Section 33: we have some qubit A described by $\mathcal{H}_A$ and some environment E in state $|e\rangle$ that evolves according to

$$|0\rangle|e\rangle \xrightarrow{D} |0\rangle|e_0\rangle$$
$$|1\rangle|e\rangle \xrightarrow{D} |1\rangle|e_1\rangle$$

  and such that $\langle e_0|e_1\rangle = ve^{i\vartheta}$ is not necessarily zero, but the $|e_i\rangle$ *are* of length 1.

- To some pure state $|\Psi\rangle = \alpha_0|0\rangle + \alpha_1|1\rangle$ we associate the density matrix

$$\rho_A = \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^* \\ \alpha_0^*\alpha_1 & |\alpha_1|^2 \end{pmatrix}$$

  and so to the evolution $|\Psi'\rangle = \alpha_0|0\rangle|e_0\rangle + \alpha_1|1\rangle|e_1\rangle$ of the state $|\Psi\rangle|e\rangle$ under then environment we associate the density operator

$$\begin{aligned}\rho = |\alpha_0|^2\Big(|0\rangle\langle 0| \otimes |e_0\rangle\langle e_0|\Big) + \alpha_0\alpha_1^*\Big(|0\rangle\langle 1| \otimes |e_0\rangle\langle e_1|\Big) \\ +\alpha_0^*\alpha_1\Big(|1\rangle\langle 0| \otimes |e_1\rangle\langle e_0|\Big) + |\alpha_1|^2\Big(|1\rangle\langle 1| \otimes |e_1\rangle\langle e_1|\Big).\end{aligned}$$

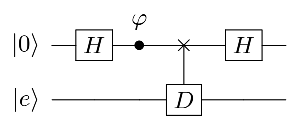
- Tracing ρ over the environment gives us the density matrix

$$\operatorname{tr}_E \rho = \begin{pmatrix} |\alpha_0|^2 & \alpha_0\alpha_1^*\langle e_1|e_0\rangle \\ \alpha_0^*\alpha_1\langle e_0|e_1\rangle & |\alpha_1|^2 \end{pmatrix}$$

- We can interpret the density operator as follows: when $\langle e_0|e_1\rangle$ tend to 0 (i.e. when the environment learns more and more) we *lose* the off-diagonal (coherence) entries.

37.6.2 Hadamard-phase-Hadamard

- We revisit the specific example found in Section 33 but in a slightly neater way, with the following circuit:



  where D is the controlled-unitary gate corresponding to the environmental decoherence.
- We follow the evolution of *just the qubit* in terms of its density matrix, moving the phase terms into the environment (this is more clearly seen when we write out what happens to the environment as well):

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \xrightarrow{H} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \xrightarrow{P_\varphi} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-i\varphi} \\ \frac{1}{2}e^{i\varphi} & \frac{1}{2} \end{pmatrix} \xrightarrow{D} \begin{pmatrix} \frac{1}{2} & \frac{1}{2}e^{-i\varphi} \\ \frac{1}{2}e^{i\varphi} & \frac{1}{2} \end{pmatrix} \xrightarrow{H} \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix}$$

- But tracing over E results in

$$\frac{1}{2}\begin{pmatrix} 1 + v\cos(\varphi+\vartheta) & iv\sin(\varphi+\vartheta) \\ -iv\sin(\varphi+\vartheta) & 1 - v\cos(\varphi+\vartheta) \end{pmatrix}$$

- Recall that the probability of measuring the output to be in state $|0\rangle$ is given by $\operatorname{tr} P_0\rho$ where $P_0 = |0\rangle\langle 0|$. As matrices, this is

$$\frac{1}{2}\operatorname{tr}\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 + v\cos(\varphi+\vartheta) & iv\sin(\varphi+\vartheta) \\ -iv\sin(\varphi+\vartheta) & 1 - v\cos(\varphi+\vartheta) \end{pmatrix} = \frac{1}{2}\big(1 + v\cos(\varphi+\vartheta)\big)$$

  which agrees with our answer in Section 33.

37.7 Completely-positive maps

- We have stated that *pure* states evolve under *unitary* evolutions:

$$\begin{aligned}|\Psi\rangle &\to U|\Psi\rangle \\ |\Psi\rangle\langle\Psi| &\to U|\Psi\rangle\langle\Psi|U^\dagger\end{aligned}$$

but general density operators *don't have to*. For example, consider the unitary evolution

$$\rho_A \otimes \rho_B \to U(\rho_A \otimes \rho_B)U^\dagger.$$

When we trace over subsystem B we obtain the map

$$\rho_A \to \mathrm{tr}_B \left(U(\rho_A \otimes \rho_B)U^\dagger \right)$$

and there is no reason why this map should necessarily be unitary.

- A good example of a *non-admissible* map is the **transpose**: $t\colon \rho \to \rho^t$ because it is not 'physically legal': consider the state $|\Psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle|0\rangle + |1\rangle|1\rangle)$. Then the transpose acts on $\rho = |\Psi\rangle\langle\Psi|$ as follows (*recalling that we are working in a tensor product of spaces*):

$$\rho = \left(\begin{array}{cc|cc} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{array}\right) \to \left(\begin{array}{cc|cc} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ \hline 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right) = \rho^t$$

  but ρ^t has eigenvalues ± 1 and so is *not* positive semi-definite and thus *not* a density matrix. Thus t is *not* a physically legal evolution, *even though* it acts on matrices in a way that preserves trace and positivity, because it doesn't act 'properly' with tensor products.

- The maps that we want to study are **completely-positive maps**: maps S that are trace-preserving and positive *and such that* the **extension** $S \otimes \mathbb{1}$ is also trace-preserving and positive.