Slice categories and limits

(Throughout, let I be a small category and $\mathcal{C}$ a locally small category.)

The <u>diagonal functor</u> Δ is defined by

$$\Delta : \mathcal{C} \longrightarrow \mathrm{Fun}(I, \mathcal{C})$$

$$c \longmapsto \left\{ \begin{array}{l} i \mapsto c \\ (i \to j) \mapsto (c \xrightarrow{id} c) \end{array} \right\}$$

$$(c \to d) \longmapsto \left\{ \begin{tikzcd} i \arrow[r, mapsto] \arrow[d] & c \arrow[d] \\ i \arrow[r, mapsto] & d \end{tikzcd} \right\}$$

We can then define the <u>limit functor</u>

$$\lim_I : \mathrm{Fun}(I, \mathcal{C}) \longrightarrow \mathcal{C}$$

as the right adjoint to the diagonal functor:

$$\Delta \dashv \lim_I .$$

We can also define the limit by its <u>universal property</u>:

Let $F \in \mathrm{Fun}(\mathcal{I}, \mathcal{C})$. Then the <u>limit</u> is an object $\ell \in \mathcal{C}$ along with morphisms $\{f_i : \ell \to F(i)\}_{i \in \mathcal{I}}$ such that

$$\begin{tikzcd} & F(i) \arrow[d, "F(f)"] \\ \ell \arrow[ur, "f_i"] \arrow[r, "f_j"'] & F(j) \end{tikzcd}$$

commutes for all $f : i \to j$ in $\mathcal{I}$, and such that it is <u>terminal</u> amongst such things, i.e. if $(\ell' \in \mathcal{C}, \{f'_i : X \to F(i)\}_{i \in \mathcal{I}})$ satisfies the same property then there exists a unique morphism $\ell' \to \ell$ such that

$$\begin{tikzcd} & & F(i) \arrow[d, "F(f)"] \\ \ell' \arrow[urr, bend left, "f'_i"] \arrow[r] \arrow[drr, bend right, "f'_j"'] & \ell \arrow[ur, "f_i"] \arrow[dr, "f_j"'] & F(j) \\ & & \end{tikzcd}$$

commutes for all $f : i \to j$ in $\mathcal{I}$.

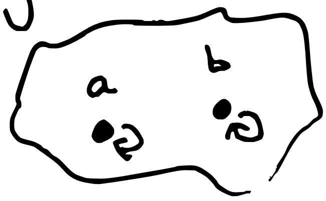
Now we can define <u>products</u> and <u>equalisers</u> in a neat abstract way using limits:

- the <u>product</u> of two objects $c, d \in \mathcal{C}$ is the limit of the functor

(we can generalise to any <u>finite</u> number of objects)

$$\begin{aligned} P : I &\longrightarrow \mathcal{C} \\ a &\longmapsto c \\ b &\longmapsto d \end{aligned}$$

where I is the discrete category with two objects, i.e.

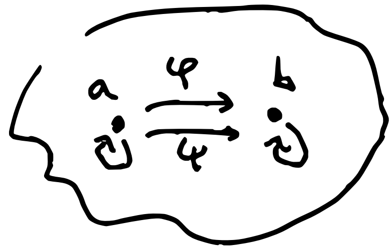


$$\mathrm{Ob}\, I = \{a, b\}$$

$$\mathrm{Hom}_I(i,j) = \begin{cases} \{id_i\} & \text{if } i = j \\ \emptyset & \text{otherwise} \end{cases}$$

- the <u>equaliser</u> of two morphisms $f, g : c \to d$ in $\mathcal{C}$ is the limit of the functor

$$\begin{aligned} E : I &\longrightarrow \mathcal{C} \\ a &\longmapsto c \\ b &\longmapsto d \\ \varphi &\longmapsto f \\ \psi &\longmapsto g \end{aligned}$$



where $\mathrm{Ob}\, I = \{a, b\}$

$$\mathrm{Hom}_I(i,j) = \begin{cases} \{\varphi, \psi\} & \text{if } i = a, j = b \\ \{id_i\} & \text{if } i = j \\ \emptyset & \text{otherwise (i.e. if } i = b, j = a) \end{cases}$$

Something that is often useful to work with is the <u>slice category</u> $\mathcal{C}/x$ (for some fixed object $x \in \mathcal{C}$):

$$\mathrm{Ob}\ \mathcal{C}/x = \{ (c,f) \mid c \in \mathcal{C},\ f \in \mathrm{Hom}_{\mathcal{C}}(c,x) \}$$

$$\mathrm{Hom}_{\mathcal{C}/x}((c,f),(d,g)) = \left\{ h \in \mathrm{Hom}_{\mathcal{C}}(c,d) \;\middle|\; \begin{tikzcd} c \arrow[d, "h"'] \arrow[dr, "f"] & \\ d \arrow[r, "g"'] & x \end{tikzcd} \text{ commutes} \right\}$$

<u>Claim</u>. If $\mathcal{C}$ has all small limits, then $\mathcal{C}/x$ also has all small limits.

<u>Proof</u>. We are going to use the very useful lemma that tells us that "if a category has all products and equalizers then it has all small colimits".

$\boxed{\text{Products}}$ Let $(c,f), (d,g) \in \mathcal{C}/x$. Since $\mathcal{C}$ has all small limits, the product $\left(c \times d, \left\{ \begin{array}{l} \pi_1 : c \times d \to c \\ \pi_2 : c \times d \to d \end{array} \right\}\right)$ (<u>in $\mathcal{C}$</u>) exists. We claim that the <u>equalizer</u>

$$\mathrm{eq}\left(c \times d \underset{g \circ \pi_2}{\overset{f \circ \pi_1}{\rightrightarrows}} x \right)$$

gives the product in $\mathcal{C}/x$.

How can we come up with this potential solution? Well, it's sort of the only option! Since products in $\mathcal{C}$ exist, it seems possible that $(c \times d)$ will be part of the product object in $\mathcal{C}/x$, but we need a morphism

$$c \times d \longrightarrow x$$

in order to get an object of $\mathcal{C}/x$. But it will need to be such that

$$\begin{tikzcd} & c \times d \arrow[dl, "\pi_1"'] \arrow[dr, "\pi_2"] \arrow[dd] & \\ c \arrow[dr, "f"'] & & d \arrow[dl, "g"] \\ & x & \end{tikzcd}$$

commutes in order for $f \circ \pi_1$ and $g \circ \pi_2$ (which are the "obvious" choices for the product morphisms) to be morphisms in $\mathcal{C}/x$.

Now we have no way of constructing such a morphism, <u>but</u> the equalizer will give us the next best thing:

$$\begin{tikzcd} & & c \arrow[dr, "f"] & \\ e \arrow[r, "\varepsilon"] & c \times d \arrow[ur, "\pi_1"] \arrow[dr, "\pi_2"'] & & x \\ & & d \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, <u>and</u> is terminal amongst such things.

But this is exactly saying that

$$\begin{tikzcd} & c \arrow[dr, "f"] & \\ e \arrow[ur, "\pi_1 \circ \varepsilon"] \arrow[dr, "\pi_2 \circ \varepsilon"'] & & x \\ & d \arrow[ur, "g"'] & \end{tikzcd}$$

commutes, <u>and</u> is terminal amongst such things, i.e. is a product!

$\boxed{\text{Equalisers}}$ Let $(c,f) \overset{h_1}{\underset{h_2}{\rightrightarrows}} (d,g)$ be a pair of morphisms in $\mathcal{C}/x$.

Then their equaliser in $\mathcal{C}/x$ would be some

$$(a \in \mathcal{C}, \alpha \in Hom_{\mathcal{C}}(a,x))$$

along with a morphism $i : a \to c$ such that

$$\underbrace{\begin{tikzcd} a \arrow[d, "i"'] \arrow[dr, "\alpha"] & \\ c \arrow[r, "f"'] & x \end{tikzcd}}_{\text{definition of a morphism in } \mathcal{C}/x} \quad \text{and} \quad \underbrace{a \xrightarrow{i} c \overset{h_1}{\underset{h_2}{\rightrightarrows}} d}_{\text{definition of an equaliser}}$$

both commute. The second diagram leads us to try taking $(a,i) = eq(c \overset{h_1}{\underset{h_2}{\rightrightarrows}} d)$ (in $\mathcal{C}$).

Then we just need to see if we can find some suitable $\alpha : a \longrightarrow x$, and we have three "obvious" options:

1) $a \xrightarrow{i} c \xrightarrow{f} x$

2) $a \xrightarrow{i} c \xrightarrow{h_1} d \xrightarrow{g} x$

3) $a \xrightarrow{i} c \xrightarrow{h_2} d \xrightarrow{g} x$

but these are all the same, by the commutativity of the two diagrams from before, so take e.g.

$$\Big(\underbrace{(a,\ f \circ i)}_{\substack{\text{an object} \\ \text{of } \mathcal{C}/x}},\ \underbrace{i}_{\substack{\text{a morphism} \\ \text{in } \mathcal{C}/x}} \Big).$$

$\square$

Something that is actually simpler to show is the dual statement for $\mathcal{C}/x$, namely that "if $\mathcal{C}$ has all small colimits then $\mathcal{C}/x$ also has all small colimits."

(by this I mean that we can prove it directly)

Proof. Let $F : I \longrightarrow \mathcal{C}/x$, and write $F(i) = (\underbrace{c_i}_{\text{an object of } \mathcal{C}}, \underbrace{\gamma_i : c_i \to x}_{\text{a morphism in } \mathcal{C}/x})$.

Since $\mathcal{C}$ has all small colimits, the induced functor

$$\begin{aligned} \tilde{F} : I &\longrightarrow \mathcal{C} \\ i &\longmapsto c_i \end{aligned}$$

has a colimit

$$\operatorname{colim}_I \tilde{F} = (\underbrace{d}_{\text{an object of } \mathcal{C}}, \underbrace{\{\iota_i : c_i \to d\}_{i \in I}}_{\text{morphisms in } \mathcal{C}})$$

where

$$\begin{tikzcd} c_i \arrow[d, "\tilde{F}(f)"'] \arrow[dr, "\iota_i"] & \\ c_j \arrow[r, "\iota_j"'] & d \end{tikzcd}$$

commutes for all $f : i \to j$ in I, and, if $(d', \{\iota'_i : c_i \to d'\}_{i \in I})$ is any other such thing, then there exists a unique morphism $\varphi : d \to d'$ such that

$$\begin{tikzcd} c_i \arrow[d, "\tilde{F}(f)"'] \arrow[dr, "\iota_i"] \arrow[drr, bend left, "\iota'_i"] & & \\ c_j \arrow[r, "\iota_j"'] \arrow[rr, bend right, "\iota'_j"'] & d \arrow[r, "\varphi"] & d' \end{tikzcd}$$

commutes.

But $(x, \{\gamma_i : c_i \to x\}_{i \in I})$ is exactly such a thing! This means that we get a unique morphism $\varphi : d \to x$ such that

$$\begin{tikzcd}
c_i \arrow[dr, "\iota_i"] \arrow[drr, bend left, "\gamma_i"] \arrow[dd, "\tilde{F}(f)"'] & & \\
& d \arrow[r, "\varphi"] & x \\
c_j \arrow[ur, "\iota_j"'] \arrow[urr, bend right, "\gamma_j"'] & &
\end{tikzcd}$$

commutes for all $f : i \to j$ in I. So

$$\Big(\underbrace{(d, \varphi : d \to x)}_{\text{an object of } \mathcal{C}/x}, \underbrace{\{\iota_i : c_i \to d\}_{i \in I}}_{\text{morphisms in } \mathcal{C}/x} \Big)$$

(since
$$\begin{tikzcd}
c_i \arrow[r, "\gamma_i"] \arrow[d, "\iota_i"'] & x \\
d \arrow[ur, "\varphi"'] &
\end{tikzcd}$$
commutes)

is a colimit in $\mathcal{C}/x$.