Let $\widetilde{\mathrm{FSet}}$ denote the groupoid of non-empty finite sets (with morphisms given by bijections), and let Vect denote the category of finite-dimensional vector spaces (over some field $k$).

Define the category of <u>symmetric sequences</u> to be the functor category

$$\mathrm{Vect}^{\Sigma} := \mathrm{Fun}(\widetilde{\mathrm{FSet}}, \mathrm{Vect}).$$

Define the functor

$$F : \mathrm{Vect}^{\Sigma} \longrightarrow \mathrm{Vect}$$

to be evaluation on the one-element set.

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<u>Construct a left adjoint to F.</u>

<u>Solution.</u> $L : \mathrm{Vect} \to \mathrm{Vect}^{\Sigma}$

$$V \longmapsto \begin{cases} \{1\} \longmapsto V \\ \{1, \dots, n\} \mapsto \{0\} \text{ for } n \geq 2 \end{cases}$$

... ok, but <u>why</u>?

By definition, we want $L: \mathrm{Vect} \to \mathrm{Vect}^{\Sigma}$ such that

$$\mathrm{Hom}_{\mathrm{Vect}^{\Sigma}}(LV, X) \cong \mathrm{Hom}_{\mathrm{Vect}}(V, FX)$$

(naturally, for all $V \in \mathrm{Vect}$ and $X \in \mathrm{Vect}^{\Sigma}$).

Now $FX := X(\{*\})$, and so we want

$$\mathrm{Hom}_{\mathrm{Vect}^{\Sigma}}(LV, X)$$

to be in bijection with

$$\mathrm{Hom}_{\mathrm{Vect}}(V, X(\{*\})).$$

Since $\mathrm{Vect}^{\Sigma}$ is a functor category, morphisms $LV \longrightarrow X$ consist of morphisms

$$LV(S) \to X(S)$$

for all $S \in \widetilde{\mathrm{FSet}}$, subject to the condition that

$$\begin{tikzcd} LV(S) \arrow[r] \arrow[d] & X(S) \arrow[d] \\ LV(T) \arrow[r] & X(T) \end{tikzcd}$$

commutes for all $S \to T$ in $\widetilde{\mathrm{FSet}}$. <u>But</u> $\widetilde{\mathrm{FSet}}$ is such that $\mathrm{Hom}_{\widetilde{\mathrm{FSet}}}(S, T) = \emptyset$ if $|S| \neq |T|$.

(using a skeletal replacement — see next page)

This means that we have no naturality condition: a morphism $LV \to X$ is exactly a collection of morphisms $LV(s) \to X(s)$ (in Vect) for all $s \in \widetilde{FSet}$.

Back to the desired bijection:

$$\mathrm{Hom}_{\mathrm{Vect}^\Sigma}(LV, X) \cong \mathrm{Hom}_{\mathrm{Vect}}(V, FX)$$

$$\mathrm{Hom}_{\mathrm{Vect}}(V, FX) = \{V \to X(\{1\})\}$$

$$\mathrm{Hom}_{\mathrm{Vect}^\Sigma}(LV, X) = \left\{ \left(LV(\{1,\dots,n\}) \to X(\{1,\dots,n\})\right)_{n \in \mathbb{N}} \right\}$$

$$= \left\{ \left( LV(\{1\}) \to X(\{1\}),\ LV(\{1,2\}) \to X(\{1,2\}), \dots \right) \right\}$$

So if we take $LV(\{1\}) = V$ and $LV(\{1,\dots,n\}) = 0$ (for $n \geq 2$) then we get a bijection

$$\mathrm{Hom}_{\mathrm{Vect}^\Sigma}(LV, X) \longleftrightarrow \mathrm{Hom}_{\mathrm{Vect}}(V, FX)$$

$$\begin{aligned}
&(\varphi : LV(\{1\}) = V \to X(\{1\}), && \longleftrightarrow \varphi : V \to FX(\{1\}) \\
&\ 0 : LV(\{1,2\}) = 0 \to X(\{1,2\}), && \\
&\ \dots && \\
&\ 0 : LV(\{1,\dots,n\}) = 0 \to X(\{1,\dots,n\}), \dots) && \\
&= (\varphi, 0, 0, 0, \dots) &&
\end{aligned}$$

$\square$

Ok, but why are these things called "symmetric sequences"?

Well we can use a skeletal replacement for $\widetilde{FSet}$, i.e.

$$\widetilde{FSet} = \left\{ \begin{array}{l} \text{objects: } S \in Set \text{ s.t. } |S| \in \mathbb{N} \\ \text{morphisms: bijections} \end{array} \right\}$$

is equivalent to

(symmetric group on $n$ elements)

$$\Sigma = \left\{ \begin{array}{l} \text{objects: } n \in \mathbb{N} \\ \text{morphisms: } Hom_\Sigma(m,n) = \begin{cases} S_n & \text{if } m=n \\ \phi & \text{if } m \neq n \end{cases} \end{array} \right\}$$

Then a functor

$$X : \Sigma \longrightarrow Vect$$

is exactly given by

- a vector space $V_n = X(n)$ for all $n \in \mathbb{N}$
- a morphism $V_n \xrightarrow{X(\sigma)} V_n$ for all $\sigma \in S_n$ (and we call such a thing an <u>action of $S_n$ on $V_n$</u>).

i.e. a functor $X \in Fun(\Sigma, Vect) = Vect^\Sigma$ (standard notation: $\mathcal{D}^\mathcal{C} := Fun(\mathcal{C}, \mathcal{D})$) is a sequence of vector spaces endowed with $S_n$-actions!

(Even simpler if you replace Vect with $\left\{ \begin{array}{l} \text{objects: } n \in \mathbb{N} \\ \text{morphisms: } Hom(m,n) = n \times m \text{ matrices} \end{array} \right\}$ which is equivalent to Vect, but skeletal)