$\mathrm{Tor}^{\mathbb{Z}}$ and $\mathrm{Ext}_{\mathbb{Z}}$ cheat sheet

Since abelian groups are exactly $\mathbb{Z}$-modules, we can calculate Tor and Ext of them.

Fact. When $R=\mathbb{Z}$, and considering only finitely generated modules, ⚠

- projective R-module = free abelian group
- injective R-module = divisible abelian group

(• flat R-module = torsion-free abelian group)

(N.B. projective $\Rightarrow$ flat)

in general, free $\Rightarrow$ projective $\Rightarrow$ flat
↑ this is an $\Leftrightarrow$ when we look at f.g. $\mathbb{Z}$-mod.

Recall that (for any R)

$\mathrm{Tor}(M,N) \cong \mathrm{Tor}(N,M)$ { M is flat $\Longrightarrow \mathrm{Tor}_n(M,-) = \mathrm{Tor}_n(-,M) = 0$ for all $n \geq 1$
(so, in particular, if M is projective)

$\mathrm{Ext}(M,N) \not\cong \mathrm{Ext}(N,M)$ { M is projective $\Rightarrow \mathrm{Ext}^n(M,-) = 0$ for all $n \geq 1$
M is injective $\Rightarrow \mathrm{Ext}^n(-,M) = 0$ for all $n \geq 1$

Going back to (finitely-generated) abelian groups

Fact. $\mathrm{Tor}_n = \mathrm{Ext}^n = 0$ for $n \geq 2$

So we'll write $\mathrm{Tor} = \mathrm{Tor}_1$ and $\mathrm{Ext} = \mathrm{Ext}_1$.

$\boxed{\mathrm{Tor}}$ ① $\mathrm{Tor}(-,\mathbb{Z}) = \mathrm{Tor}(\mathbb{Z},-) = 0$
② $\mathrm{Tor}(-,\mathbb{Q}) = \mathrm{Tor}(\mathbb{Q},-) = 0$

$\mathbb{Z}$ and $\mathbb{Q}$ are both torsion-free and thus flat

③ $\mathrm{Tor}(A, \mathbb{Z}/m\mathbb{Z}) \cong \mathrm{Tor}(\mathbb{Z}/m\mathbb{Z}, A)$
$\cong \{a \in A \mid ma = 0\}$

use the projective resolution $\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}$ of $\mathbb{Z}/m\mathbb{Z}$

④ $\mathrm{Tor}(A, \mathbb{Q}/\mathbb{Z}) \cong \mathrm{Tor}(\mathbb{Q}/\mathbb{Z}, A)$
$\cong \mathrm{tors}(A)$

use the projective resolution $\mathbb{Z} \hookrightarrow \mathbb{Q}$ of $\mathbb{Q}/\mathbb{Z}$

Structure theorem for f.g. abelian groups: $A \cong \mathbb{Z}^r \oplus \mathrm{tors}(A)$

$\boxed{\mathrm{Ext}}$ ① $\mathrm{Ext}(\mathbb{Z}, B) = 0$ ($\mathbb{Z}$ is projective)
② $\mathrm{Ext}(A, \mathbb{Q}) = 0$ ($\mathbb{Q}$ is injective)
③ $\mathrm{Ext}(\mathbb{Z}/m\mathbb{Z}, B) \cong B/mB$

use the projective resolution $\mathbb{Z} \xrightarrow{\cdot m} \mathbb{Z}$ of $\mathbb{Z}/m\mathbb{Z}$