Vector spaces as algebras over a monad

Question: Can we construct $\text{Vect}_{\mathbb{R}}$ (the category of real vector spaces) as the category of T-algebras for some monad $T: \text{Set} \to \text{Set}$?

Answer: yes!

Recall that an adjunction $F: \mathcal{C} \underset{\perp}{\rightleftarrows} \mathcal{D} : G$ gives us a monad GF on $\mathcal{C}$.

For $\text{Vect}_{\mathbb{R}}$ and Set, we have the adjunction

$$\text{Free} : \text{Set} \underset{\perp}{\rightleftarrows} \text{Vect}_{\mathbb{R}} : \text{Forget}$$

where Forget is the forgetful functor that simply takes the underlying set, and Free sends any set X to the free vector space on X (i.e. the vector space given by the basis $\{x\}_{x \in X}$). This gives us a monad

$$T = \text{Forget} \circ \text{Free} : \text{Set} \to \text{Set}$$

(with $\eta : id_{set} \Rightarrow T$ and $\mu : T^2 \Rightarrow T$ induced by the unit and counit (respectively) of the adjunction)

What is an algebra for this monad?

By definition, an algebra $X \in \mathrm{Alg}_T(\mathrm{Set})$ is a set X along with a function

$$\alpha : TX = \mathrm{Forget}(\mathrm{Free}(X)) \longrightarrow X.$$

Our claim is that this is exactly the data of a vector space structure on X. How so?

Well, given $x, y \in X$, we know that $x+y \in \mathrm{Free}(X)$, and thus $x+y \in \mathrm{Forget}(\mathrm{Free}(X))$. So we define

$$x+y := \alpha(x+y),$$

and, similarly,

$$\lambda x := \alpha(\lambda x).$$

(these are written misleadingly, since the x on the left is an element on X, but the one on the right is an element of $\mathrm{Forget}(\mathrm{Free}(X))$!)

The fact that the vector space axioms are satisfied follows from the fact that α satisfies the algebra axioms, e.g.

$$\begin{array}{ccc} TTX & \xrightarrow{T\alpha} & TX \\ {\scriptstyle \mu_X}\downarrow & \circlearrowleft & \downarrow{\scriptstyle \alpha} \\ TX & \xrightarrow[\alpha]{} & X \end{array}$$

tells us (amongst other things) that $(x+y)+z = x+(y+z)$.

(to understand this, you need to make sure that you understand μ and η!)